Stability conditions of fractional discrete-time scalar systems with pure delay

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Abstract: In the paper the problem of stability of fractional discrete-time linear scalar systems with state space pure delay is considered. Using the classical D-decomposition method, the necessary and sufficient condition for practical stability as well as the sufficient condition for asymptotic stability are given.

Keywords: asymptotic stability, practical stability, fractional order, discrete-time linear system

1. Introduction

In recent years considerable attention has been paid to fractional calculus and its application in many areas of science and engineering such as: control systems, electrical engineering, mechanics, chemistry, biology, signal and image processing. Fractional differentiation is used in modelling many physical phenomena, where similar modelling with the traditional (integer) differentiation either fails or provides poor results. A variety of fractional models can be found in various fields such as diffusion, fluid flow, turbulence, viscoelasticity, electric networks, polymer physics and propagation of seismic waves. State of the art of fractional systems and the application of fractional differentiation have been presented in monographs (e.g., [1, 11–14, 16, 17, 20, 21]) and papers (e.g., [5, 7, 22]).

The fundamental matter in the dynamic systems theory is the stability problem. In case of the linear continuous-time fractional systems this problem has been considered in many publications (e.g., [2, 11, 12, 15, 18]). On the other hand, a stability problem of the linear discrete-time fractional systems is more complicated and less advanced. It results from the fact that the asymptotic stability of such systems corresponds to the asymptotic stability of the associated infinite dimensional discrete-time systems with delays. In practice the number of delays is limited by the so-called length of practical implementation and the discrete-time system with finite number of delays is obtained. Its asymptotic stability is the so-called practical stability of the fractional discrete-time system. The problem of practical stability of fractional discrete-time systems has been considered in [3, 4, 10, 11, 12] for positive systems and in [3, 6, 7, 9] for standard systems (non-positive).

Main purpose of this paper is to establish new stability conditions for the fractional discrete-time linear scalar system with pure delay. The practical stability and asymptotic stability will be analysed. New necessary and sufficient condition for practical stability and the sufficient condition for asymptotic stability will be proposed.

2. Problem formulation

Consider the fractional discrete-time linear scalar systems with state space pure delay, described by the homogeneous equation

$$\Delta^{\alpha} x_{i+1} = a_1 x_{i-1}, \quad \alpha \in (0,\ 1), \quad i \in Z_+ \tag{1}$$

with the initial condition x_{-l} $(l = 0, 1)$, where $x_i \in \Re$ is the state vector and a_1 is the scalar.

In this paper the following definition of the fractional difference [10–12] will be used

$$\Delta^{\alpha} x_i = \sum_{k=0}^{i} (-1)^k \binom{\alpha}{k} x_{i-k}, \tag{2}$$

where $\alpha \in \Re$ is the order of the fractional difference, and

$$\binom{\alpha}{k} = \frac{\alpha!}{k!(\alpha-k)!}. \tag{3}$$

Using definition (2), after transformation, equation (1) can be written in the form

$$x_{i+1} = \alpha x_i + a_1 x_{i-1} + \sum_{k=1}^{i} c_k(\alpha) x_{i-k}, \tag{4}$$

where

$$c_k(\alpha) = (-1)^k \binom{\alpha}{k+1}, \quad k = 1, 2, \ldots \tag{5}$$

The coefficients (5) can be easily calculated using the following formula [4]

$$c_{k+1}(\alpha) = c_k(\alpha)\frac{k+1-\alpha}{k+2}, \quad k = 1, 2, \ldots \tag{6}$$

with $c_1(\alpha) = 0.5\alpha(1-\alpha)$.

Note that equation (4) represents a linear discrete-time system with a growing number of delays in state.

From (6) it follows that the coefficients $c_k(\alpha)$ are positive for $\alpha \in (0,\ 1)$ and decrease rapidly with an increase of k. Therefore, we can assume that the value of k in equation (4) may be limited by some natural number L. This number is called the length of the practical implementation [10]. In this case equation (4) can be written in the form

$$x_{i+1} = \alpha x_i + a_1 x_{i-1} + \sum_{k=1}^{L} c_k(\alpha) x_{i-k}, \quad i \in Z_+. \tag{7}$$

Equation (7) represents a linear discrete-time system with L delays in state. Moreover, the system (7) is called the practical realization of fractional system (1).

The definition of practical stability and the related definition of asymptotic stability for fractional discrete-time systems have been introduced in the work [10]. With regard to the system (1) these definitions take the following forms.

Definition 1. The fractional system (1) is called practically stable if the system (7) is asymptotically stable.

Definition 2. The fractional system (1) is called asymptotically stable if the system (7) is practically stable for $L \to \infty$.

Using the stability theory of discrete-time linear systems and Definition 1 we obtain the following theorem.

Theorem 1. The fractional system (1) with given length L of practical implementation is practically stable if and only if

$$w(z) \neq 0, \quad |z| \geqslant 1, \tag{8}$$

where

$$w(z) = z - \alpha - a_1 z^{-1} - \sum_{k=1}^{L} c_k(\alpha) z^{-k} \tag{9}$$

is the characteristic polynomial of the system (7).

The characteristic equation $w(z) = 0$ of the system (7) can be written as

$$z^{L+1} - \alpha z^{L} - a_1 z^{L-1} - \sum_{k=1}^{L} c_k(\alpha) z^{L-k} = 0. \tag{10}$$

Well-known methods for testing the asymptotic stability of discrete-time systems can be used to study the practical stability of the fractional system (1) (asymptotic stability of system (7)). This is not an easy task in the case of high degree of equation (10), which depends on the length L of practical implementation.

The main aim of this paper is to give new necessary and sufficient condition for practical stability and new necessary condition for asymptotic stability of the system (1), which do not require direct checking of condition (8). The proposed stability conditions do not require a priori knowledge of the characteristic polynomial (9).

3. Solution of the problem

In the asymptotic stability analysis of fractional discrete-time system (1) we consider, without reducing generality of considerations, the system described by the equation

$$\Delta^{\alpha} x_{i+1} = (a_1 + jb) x_{i-1}, \quad j^2 = -1, \quad \alpha \in (0,\, 1), \tag{11}$$

where a_1 and b are real numbers.

For the system (11) equation (7) takes the form

$$x_{i+1} = \alpha x_i + (a_1 + jb) x_{i-1} + \sum_{k=1}^{L} c_k(\alpha) x_{i-k}. \tag{12}$$

The characteristic polynomial of the system (12) is the polynomial with complex coefficients of the form

$$\tilde{w}(z) = z - \alpha - (a_1 + jb) z^{-1} - \sum_{k=1}^{L} c_k(\alpha) z^{-k}. \tag{13}$$

The D-decomposition method [8, 18, 19] will be applied for analysis of stability of the system (12) in connection with values of the parameters a_1 and b. Using this method, the stability region in the parameter plane $(a_1,\, b)$ may be determined and the parameters can be specified. The plane $(a_1,\, b)$ is decomposed by the so-called boundaries of D-decomposition into finite number of regions D(q). The polynomial (13) for any point in the region D(q) has q zeros which satisfy the condition $|z| > 1$. The stability region of polynomial (13) is the region denoted as D(0). For any point in the D-decomposition boundaries the polynomial (13) has at least one zero on the unit circle in the complex z-plane. Those zeros may be real or complex, thus, we have the real zero boundary or the complex zero boundary. Any point in the real zero boundary corresponds to such values of a_1 and b for which the polynomial (13) has zeros $z = 1$ or $z = -1$, while any point in the complex zero boundary corresponds to such values of a_1 and b for which the polynomial (13) has complex zeros satisfying the condition $|z| = 1$.

Firstly, the real zero boundary will be obtained. For $z = 1$ and $z = -1$ from the equation $\tilde{w}(z) = 0$ after transformation we get, respectively,

$$a_1 + jb = 1 - \alpha - \sum_{k=1}^{L} c_k(\alpha), \tag{14}$$

$$a_1 + jb = 1 + \alpha + \sum_{k=1}^{L} c_k(\alpha)(-1)^{-k}. \tag{15}$$

Hence, in the plane $(a_1,\, b)$ the real zero boundaries are two points: the point corresponding to $z = 1$ with coordinates

$$a_1 = 1 - \alpha - \sum_{k=1}^{L} c_k(\alpha), \quad b = 0, \tag{16}$$

and the point corresponding to $z = -1$ with coordinates

$$a_1 = 1 + \alpha + \sum_{k=1}^{L} c_k(\alpha)(-1)^{-k}, \quad b = 0. \tag{17}$$

Now, the complex zero boundary will be determined by solving the following complex equation with respect to a_1 and b

$$\begin{aligned} \tilde{w}(\exp(j\omega)) = \exp(j\omega) - \alpha - (a_1 + jb)\exp(-j\omega) \\ - \sum_{k=1}^{L} c_k(\alpha)\exp(-j\omega k) = 0. \end{aligned} \tag{18}$$

This equation is obtained after introducing substitution $z = \exp(j\omega)$, $\omega \in [0,\, 2\pi]$ (boundary of the unit circle in the complex z-plane) in the polynomial (13) and equating to 0. Finally, by solving equation (18) we get

$$\begin{aligned} a_1(\omega) = 2\cos(\omega)^2 - 1 - \alpha\cos(\omega) \\ - \sin(\omega)\sum_{k=1}^{L} c_k(\alpha)\sin(\omega k) \\ - \cos(\omega)\sum_{k=1}^{L} c_k(\alpha)\cos(\omega k), \end{aligned} \tag{19}$$

$$\begin{aligned} b(\omega) = 2\cos(\omega)\sin(\omega) - \alpha\sin(\omega) \\ - \sin(\omega)\sum_{k=1}^{L} c_k(\alpha)\cos(\omega k) \\ + \cos(\omega)\sum_{k=1}^{L} c_k(\alpha)\sin(\omega k). \end{aligned} \tag{20}$$

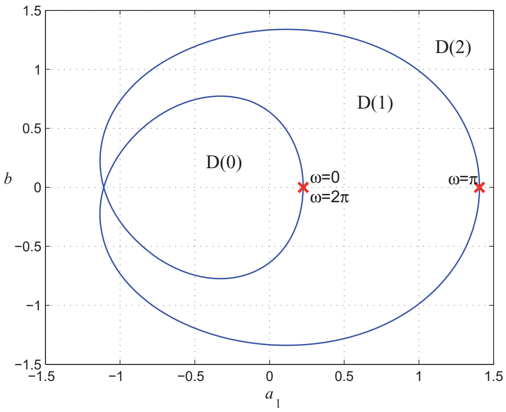

Fig. 1. The practical stability region D(0) of system (11) for $\alpha = 0.5$ and $L = 5$

Rys. 1. Obszar stabilności praktycznej D(0) układu (11) dla $\alpha = 0{,}5$ i $L = 5$

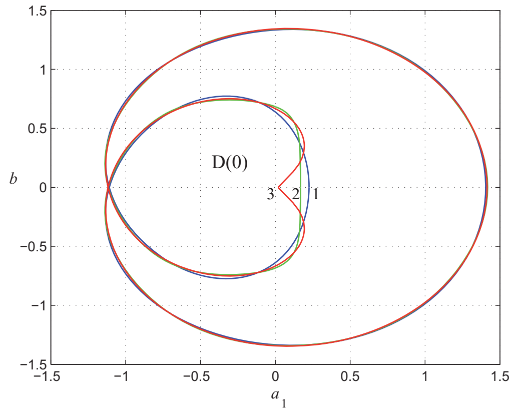

Fig. 2. The practical stability regions D(0) of system (11) for $\alpha = 0.5$ and $L = 5$ (boundary 1), $L = 10$ (boundary 2), $L = 1000$ (boundary 3)

Rys. 2. Obszary stabilności praktycznej D(0) układu (11) dla $\alpha = 0{,}5$ oraz $L = 5$ (krzywa 1), $L = 10$ (krzywa 2), $L = 1000$ (krzywa 3)

Equations (19) and (20) describe the complex zero boundary in plane (a_1, b). Note that from these equations for $\omega = 0$ and $\omega = \pi$ we obtain formulas (16) and (17), respectively.

The practical stability region of the system (11), that is, the asymptotic stability region of the system (12) for a given L, for the example values $\alpha = 0.5$ and $L = 5$, is shown in fig. 1. The complex zero boundary obtained for $\omega \in [0, 2\pi]$ divides the plane (a_1, b) into two bounded regions and one unbounded. The real zero boundary is denoted by x-marks in fig. 1. The asymptotic stability region D(0) of system (12) is chosen by testing an arbitrary point from each region and checking the asymptotic stability of the polynomial (13). For example, choosing the point with coordinates $a_1 = -0.5$ and $b = 0$ we obtain the following zeros of polynomial (13): $z_1 = -0.367$, $z_2 = 0.622$, $z_{3,4} = -0.121 \pm j0.421$, $z_{5,6} = 0.243 \pm j0.639$. For all these zeros the condition $|z| < 1$ is satisfied, thus the region with this point is the stability region D(0). Hence, in the plane (a_1, b), the practical stability region of the system (11) is the region bounded by the closed curve $a_1(\omega) + jb(\omega)$, where $a_1(\omega)$ and $b(\omega)$ are calculated from (19) and (20).

Fig. 2 shows the practical stability regions of the system (11) with $\alpha = 0.5$ and different values of L, while fig. 3 shows the practical stability regions with $L = 1000$ and different values of α. It is easy to check that for $\alpha = 0$ we obtain the unit disc.

The state equation of system (1) may be obtained after setting $b = 0$ in (11). Therefore, for the fractional system (1) the practical stability region D(0), shown in fig. 1, reduces to an interval of the real axis. The lower and upper endpoints of this interval will be denoted by $a_{1\,\min}$ and $a_{1\,\max}$. The upper endpoint $a_{1\,\max}$ corresponds to the real zero boundary (16), and its value can be calculated from the following formula

$$a_{1\,\max}(\alpha, L) = 1 - \alpha - \sum_{k=1}^{L} c_k(\alpha). \qquad (21)$$

The lower endpoint $a_{1\,\min}$ corresponds to the self-intersection point of the complex zero boundary, and its

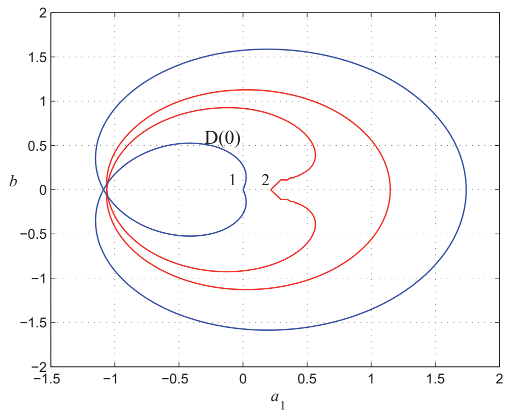

Fig. 3. The practical stability regions D(0) of system (11) for $L = 1000$ and $\alpha = 0.2$ (boundary 1), $\alpha = 0.8$ (boundary 2)

Rys. 3. Obszary stabilności praktycznej D(0) układu (11) dla $L = 1000$ oraz $\alpha = 0{,}2$ (krzywa 1), $\alpha = 0{,}8$ (krzywa 2)

value can be calculated using equations (19) and (20) according to the following procedure.

Step 1. Assuming $b = 0$ in the equation (20) compute $\omega = \omega_p \in (0, \pi)$.

Step 2. In equation (19) substitute ω_p and compute

$$a_{1\,\min}(\alpha, L) = a_1(\omega_p). \qquad (22)$$

The values of $a_{1\,\min}$ and $a_{1\,\max}$ depend on the given order $\alpha \in (0, 1)$ and the given length L of practical implementation. Fig. 2 shows that the practical stability region of the system (1) (interval of the real axis) becomes smaller for greater values of L.

On the basis of the conducted considerations the following theorem can be formulated.

Theorem 2. The fractional system (1) with the given length L of practical implementation is practically stable if and only if

$$a_{1\,\min}(\alpha, L) < a_1 < a_{1\,\max}(\alpha, L), \qquad (23)$$

where $a_{1\,\min}(\alpha, L)$ and $a_{1\,\max}(\alpha, L)$ are computed from the formulas (22) and (21).

Graphs of $a_{1\,\min}(\alpha, L)$ and $a_{1\,\max}(\alpha, L)$, given by formulas (22) and (21), as functions of the fractional order $\alpha \in (0,\ 1)$ for small $L = 10$ and large $L = 1000$ values of the length L of practical implementation, are shown in fig. 4. The practical stability region of the system (1) in the plane (α, a_1) lies between the graphs $a_{1\,\min}(\alpha, L)$ and $a_{1\,\max}(\alpha, L)$. This region for a given L determines values of a_1, for which the system (1) is practically stable with a given $\alpha \in (0,\ 1)$.

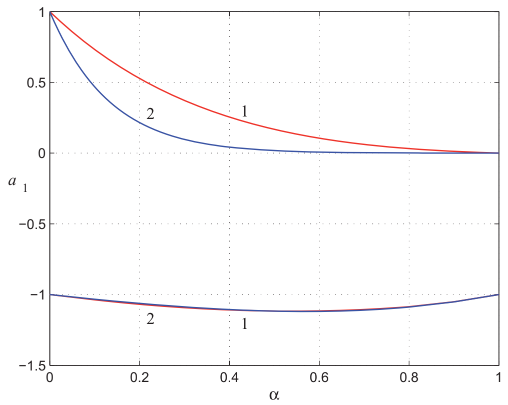

Fig. 4. The practical stability regions of system (1) in the parameter plane $(\alpha,\ a_1)$ for $L = 10$ (boundaries 1) and $L = 1000$ (boundaries 2)

Rys. 4. Obszary praktycznej stabilności układu (1) na płaszczyźnie $(\alpha,\ a_1)$ dla $L = 10$ (krzywe 1), $L = 1000$ (krzywe 2)

Fig. 4 shows that, for a fixed $\alpha \in (0,\ 1)$, values of $a_{1\,\min}(\alpha, L)$ differ slightly for small and large values of L, whereas values of $a_{1\,\max}(\alpha, L)$ differ significantly for small and large values of L.

Now we consider the problem of asymptotic stability.

To establish conditions for asymptotic stability of the system (1), the conditions (23) for $L \to \infty$ will be considered.

Using the formula [5, 10, 11]

$$\sum_{k=1}^{\infty} c_k(\alpha) = 1 - \alpha, \quad \alpha \in (0,\ 1), \tag{24}$$

from (21) for $L \to \infty$ we obtain

$$\lim_{L\to\infty} a_{1\,\max}(\alpha, L) = 0. \tag{25}$$

It is easy to check that the self-intersection point of the complex zero boundaries for all $\alpha \in (0,\ 1)$ and $L \to \infty$ lies to the left of point $a_1 = -1$ (see fig. 2 and fig. 3).

From the above we obtain the sufficient condition for asymptotic stability of the fractional discrete-time linear scalar system (1) with pure delay.

Lemma 1. If

$$-1 < a_1 < 0 \tag{26}$$

the fractional system (1) is asymptotically stable.

Example. Consider the fractional system (1) with $\alpha = 0.2$. Find values of coefficient a_1 for which the system is practically stable for $L = 10$ and $L = 1000$.

Using Theorem 2 and fig. 4 we find out that the system (1) with $\alpha = 0.2$ is practically stable for $a_1 \in (-1.069,\ 0.528)$ if $L = 10$ and for $a_1 \in (-1.063,\ 0.216)$ if $L = 1000$. In this case the system (1) with $\alpha = 0.2$ and $a_1 = 0.5$ is practically stable for $L = 10$, but it is not practically stable for $L = 1000$.

4. Concluding remarks

The problem of practical stability and asymptotic stability of discrete-time linear scalar system (1) of fractional order $\alpha \in (0,\ 1)$ with pure delay is analysed. Using the classical D-decomposition method new necessary and sufficient conditions for practical stability (Theorem 2) and new sufficient condition for asymptotic stability (Lemma 1) are established.

The work can be extended for a class of systems described by the equation $\Delta^\alpha x_{i+1} = A_1 x_{i-1}$ with diagonal state space matrix A_1.

Acknowledgements

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Warunki stabilności skalarnych układów dyskretnych niecałkowitego rzędu z czystym opóźnieniem

Streszczenie: W pracy rozpatrzono problem stabilności liniowych skalarnych układów dyskretnych niecałkowitego rzędu z czystym opóźnieniem zmiennych stanu. Wykorzystując metodę podziału D podano warunek konieczny i wystarczający praktycznej stabilności oraz warunek wystarczający stabilności asymptotycznej.

Słowa kluczowe: stabilność asymptotyczna, stabilność praktyczna, niecałkowity rząd, liniowy układ dyskretny

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