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DEKOMPOZYCJA LINIOWYCH DODATNICH UKŁADÓW DYSKRETNYCH NIECAŁKOWITEGO RZĘDU

Podana zastanie metoda dekompozycji nieosiągalnych dodatnich układów dyskretnych niecałkowitego rzędu na część osiągalną i nieosiągalną. Sformułowane i udowodnione zostaną warunki tej dekompozycji układu nieosiągalnego na część osiągalną i nieosiągalną. Zaproponowana zostanie procedura dekompozycji a jej skuteczność zostanie zilustrowane przykładami numerycznymi.

DECOMPOSITION OF THE POSITIVE FRACTIONAL DISCRETE-TIME LINEAR SYSTEM

The decomposition of unreachable positive fractional discrete-time linear systems into the reachable and unreachable parts is addressed. Conditions for the decomposition of the unreachable system into reachable and unreachable parts are established. A procedure for the decomposition is proposed and illustrated by numerical examples

1. INTRODUCTION

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive linear theory is given in the monographs [2, 3].

Mathematical fundamentals of the fractional calculus are given in the monographs [12, 13, 17] The positive fractional linear systems have been introduced in [5-7]. The stability of fractional linear systems has been investigated in [14, 16, 18] The reachability of fractional positive continuous-time linear systems has been addressed in [6].Some recent interesting results in fractional systems theory and its applications can be found in [12, 14, 16, 17, 20].

The notions of controllability and observability and the decomposition of linear systems have been introduced by Kalman [10, 11]. These notions are the basic concepts of the modern control theory [1, 8, 9, 15, 19]. They have been also extended to positive linear systems [2, 3]. In [4] the decomposition of the pairs (A, B) and (A, C) of positive discrete-time linear systems have been addressed.

In this paper the idea of Kalman's decomposition theorem will be extended to positive fractional discrete-time linear systems. Conditions will be established for the decomposition of the unreachable system into reachable and unreachable parts.

The paper is organized as follows. In section 2 the basic definitions and theorems concerning positive fractional linear systems are recalled. The main result of the paper is given in section ,where conditions for the decomposition and procedure for computation of the reachable and unreachable parts are presented. Concluding remarks are given in section 4.

The set $n\times m$ real matrices will be denoted by $\Re^{n\times m}$ and $\Re^n := \Re^{n\times 1}$. The set of $m\times n$ real matrices with nonnegative entries will be denoted by $\Re_+^{m\times n}$ and $\Re_+^n := \Re_+^{n\times 1}$. The set of non-negative integers will be denoted by Z_+ and the $n\times n$ identity matrix by I_n.

2. PRELIMINERIES

Consider the fractional discrete-time linear system [7]

$$\Delta^{\alpha} x_{i+1} = Ax_i + Bu_i \,, \; i \in Z_+ \,, \; 0 < \alpha < 1 \tag{1}$$

where α is fractional order, $x_i \in \Re^n$ is the state vector $u_i \in \Re^m$ is the input vector and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$. The fractional difference of the order $\alpha \in [0,1)$ is defined by

$$\Delta^{\alpha} x_i = x_i + \sum_{j=1}^{i} (-1)^j \binom{\alpha}{j} x_{i-j} \tag{2}$$

where

$$\binom{\alpha}{j} = \frac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} \,, \; j = 1,2,\ldots \tag{3}$$

Substitution of (2) into (1) yields

$$x_{i+1} = A_{\alpha} x_i + \sum_{j=1}^{i} c_j(\alpha) x_{i-j} + Bu_i \,, \; i \in Z_+ \tag{4}$$

where

$$A_{\alpha} = A + I_n \alpha \,, \; c_j(\alpha) = (-1)^j \binom{\alpha}{j+1} > 0 \,, \; j = 1,2,\ldots \tag{5}$$

The solution of (4) has the form [7]

$$x_i = \Phi_i x_0 + \sum_{k=0}^{i-1} \Phi_{i-k-1} B u_i \tag{6}$$

and the matrix Φ_i can be computed from the formula

$$\Phi_{i+1} = A_{\alpha} \Phi_i + \sum_{j=1}^{i} c_j(\alpha) \Phi_{i-j} \,, \; \Phi_0 = I_n \tag{7}$$

Definition 1. [7] The system (4) is called the (internally) positive fractional system if and only if $x_k \in \Re_+^n$, $k \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all input sequences $u_k \in \Re_+^m$, $k \in Z_+$.

Theorem 1. [7] The fractional system (4) is positive for $0 < \alpha < 1$ if and only if

$$A_{\alpha} \in \Re_+^{n\times n} \,, \; B \in \Re_+^{n\times m} \tag{8}$$

Definition 2. [7] The positive fractional system (1) (and (4)) is called reachable in q steps if for every given final state $x_f \in \Re_+^n$ there exists an input sequence $u_k \in \Re_+^m$ for $k = 0,1,...,q-1$ such that $x_q = x_f$.

A column of the form ae_i, $i = 1,...,n$; $a > 0$ is called monomial. A square matrix A is called monomial if in each row and in each column only one entry is positive and the remaining entries are zero.

Theorem 2. [7] The positive fractional system (1) is reachable in q steps if and only if the matrix

$$R_q = [B \quad \Phi_1 B \quad ... \quad \Phi_{q-1} B] \tag{9}$$

contains n linearly independent monomial columns.

Theorem 3. [7] The positive fractional system (1) is reachable in q steps only if the matrix $[B \quad A_{\alpha}]$ contains n linearly independent monomial columns.

3. DECOMPOSITION OF THE PAIR (A_α, B) OF POSITIVE FRACTIONAL SYSTEM

Theorem 3. The positive fractional system (1) (and (4)) is reachable in q steps if and only if the matrix

$$[B \quad \Phi_1 B] = [B \quad A_\alpha B] \in \Re_+^{n\times 2m} \tag{10}$$

contains n linearly independent monomial columns.

Proof. By Theorem 2 the positive fractional system (1) is reachable in q steps if and only if the matrix (9) contains n linearly independent monomial columns. From (7) we have

$$\Phi_{i+1}B = A_\alpha \Phi_i B + \sum_{j=1}^{i} c_j(\alpha)\Phi_{i-j}B \tag{11}$$

From (11) it follows that $\Phi_k B$ for $k = 2,3,...$ do not introduce in the matrix (9) new additional linearly independent monomial columns. Therefore, the matrix (9) contains n linearly independent columns if and only if the matrix (10) contains n linearly independent columns. □

From Theorem 3 we have the following important corollary.

Corollary 1. The positive fractional system (1) is reachable only if $n \le 2m$. Single-input system ($m = 1$) is reachable only for $n \le 2$.

Let the pair (A_α, B) of the positive fractional system (1) be unreachable but the matrix (10) contains at least one monomial column. First we shall consider the single-input ($m = 1$) system. We assume that the matrix $B \in \Re_+^n$ is monomial and the matrix (10) has $n_1 < n$ linearly independent monomial columns

$$P_1, ..., P_{n_1} \tag{12}$$

It is always possible to choose $n_2 = n - n_1$ linearly independent monomial columns

$$P_{n_1+1}, ..., P_n \tag{13}$$

which are orthogonal to the columns (12). The matrix

$$P = [P_1 \quad ... \quad P_{n_1} \quad P_{n_1+1} \quad ... \quad P_n] \tag{14}$$

is monomial and its inverse P^{-1} can be computed by transposition and substitution each positive entry by its inverse. It is assumed that the following condition

$$P_k^T A P_{n_1} = 0 \text{ for } k = n_1 + 1, ..., n \tag{15}$$

is satisfied. Note that (15) holds if $A_\alpha P_{n_1}$ is a linear combination of the monomial columns (12).

Theorem 4. Let the positive fractional system (1) be unreachable but the matrix (10) has n_1 $(n_1 < n)$ linearly independent monomial columns and let the assumption (15) be satisfied. Then the pair (A_α, B) of the system can be reduced by the similarity transformation with the monomial matrix (14) to the form

$$\bar{A} = P^{-1}A_\alpha P = \begin{bmatrix} \bar{A}_1 & \bar{A}_{12} \\ 0 & \bar{A}_2 \end{bmatrix}, \quad \bar{B} = P^{-1}B = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \tag{16}$$

where the pair

$$\bar{A}_1 \in \Re_+^{n_1\times n_1}, \quad \bar{B}_1 \in \Re_+^{n_1} \tag{17}$$

is reachable and the pair $\bar{A}_2 \in \Re_+^{n_2\times n_2}$, $\bar{B}_2 \in \Re_+^{n_2}$ is unreachable.

Proof. Taking into account that the columns (13) are orthogonal to the columns (12) and the assumption (15) we obtain

$$\bar{A} = P^{-1}A_\alpha P = P^T A_\alpha P = \begin{bmatrix} P_1^T \\ \vdots \\ P_{n_1}^T \\ P_{n_1+1}^T \\ \vdots \\ P_n^T \end{bmatrix} [A_\alpha P_1 \quad \ldots \quad A_\alpha P_{n_1} \quad A_\alpha P_{n_1+1} \quad \ldots \quad A_\alpha P_n] = \begin{bmatrix} \bar{A}_1 & \bar{A}_{12} \\ 0 & \bar{A}_2 \end{bmatrix} \tag{18a}$$

$$\begin{aligned} \bar{A}_1 &= \begin{bmatrix} P_1^T \\ \vdots \\ P_{n_1}^T \end{bmatrix} [A_\alpha P_1 \quad \ldots \quad A_\alpha P_{n_1}], \quad \bar{A}_{12} = \begin{bmatrix} P_1^T \\ \vdots \\ P_{n_1}^T \end{bmatrix} [A_\alpha P_{n_1+1} \quad \ldots \quad A_\alpha P_n], \\ 0 &= \begin{bmatrix} P_{n_1+1}^T \\ \vdots \\ P_n^T \end{bmatrix} [A_\alpha P_1 \quad \ldots \quad A_\alpha P_{n_1}], \quad \bar{A}_2 = \begin{bmatrix} P_{n_1+1}^T \\ \vdots \\ P_n^T \end{bmatrix} [A_\alpha P_{n_1+1} \quad \ldots \quad A_\alpha P_n], \end{aligned} \tag{18b}$$

and

$$\bar{B} = P^{-1}B = P^T B = \begin{bmatrix} P_1^T \\ \vdots \\ P_n^T \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \tag{19}$$

since

$$P_k^T B = \begin{cases} 1 & \text{for} \quad k = 1 \\ 0 & \text{for} \quad k = 2, \ldots, n \end{cases} \tag{20}$$

It is easy to verify that the pair $(\bar{A}_1, \bar{B}_1)$ is reachable since the, matrix

$$[\bar{B}_1 \quad \bar{\Phi}_1 \bar{B}_1 \quad \ldots \quad \bar{\Phi}_{n_1-1} \bar{B}_1] \tag{21}$$

contains n_1 linearly independent monomial columns. □

Example 1. Consider the fractional system (1) for $0 < \alpha < 1$ with the matrices

$$A = \begin{bmatrix} a_{11} & 0 & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ a_{31} & 0 & -\alpha & 0 \\ a_{41} & a_{42} & 0 & -\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (a_{ij} > 0) \tag{22}$$

The matrix

$$A_\alpha = A + I_n \alpha = \begin{bmatrix} a_{11}+\alpha & 0 & 1 & 0 \\ a_{21} & a_{22}+\alpha & 0 & 1 \\ a_{31} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{bmatrix} \in \Re_+^{4\times 4} \tag{23}$$

and the fractional system is positive. The pair (A_α, B) is unreachable since the matrix

$$[B \quad \Phi_1 B \quad \Phi_2 B \quad \Phi_3 B] = \left[B \quad A_\alpha B \quad A_\alpha{}^2 B - \binom{\alpha}{2} B \quad A_\alpha{}^3 B - \binom{\alpha}{2} A_\alpha B + \binom{\alpha}{3} B\right]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & a_{22}+\alpha & (a_{22}+\alpha)^2 - \binom{\alpha}{2} \\ 0 & 0 & 0 & 0 \\ 1 & 0 & a_{42} - \binom{\alpha}{2} & a_{42}(a_{22}+\alpha) + \binom{\alpha}{3} \end{bmatrix} \tag{24}$$

has only two linearly independent monomial columns. In this case the matrix (14) has the form

$$P = [P_1 \quad P_2 \quad P_3 \quad P_4] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{25}$$

The assumption (15) is satisfied since $P_3^T A_\alpha P_2 = 0$ and $P_4^T A_\alpha P_2 = 0$. Using (16) and (25) we obtain

$$\bar{A} = P^{-1} A_\alpha P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11}+\alpha & 0 & 1 & 0 \\ a_{21} & a_{22}+\alpha & 0 & 1 \\ a_{31} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_{42} & a_{41} & 0 \\ 1 & a_{22}+\alpha & a_{21} & 0 \\ 0 & 0 & a_{11}+\alpha & 1 \\ 0 & 0 & a_{31} & 0 \end{bmatrix} = \begin{bmatrix} \bar{A}_1 & \bar{A}_{12} \\ 0 & \bar{A}_2 \end{bmatrix} \tag{26a}$$

and

$$\bar{B} = P^{-1} B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \tag{26b}$$

The pair $(\bar{A}_1, \bar{B}_1)$ where

$$\bar{A}_1 = \begin{bmatrix} 0 & a_{42} \\ 1 & a_{22}+\alpha \end{bmatrix} \text{ and } \bar{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is reachable since

$$[\bar{B}_1 \quad \bar{\Phi}_1 \bar{B}_1] = [\bar{B}_1 \quad \bar{A}_1 \bar{B}_1] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now let us consider the multi-input $(m > 1)$ positive fractional system (1). It is assumed that matrix $B \in \Re_+^{n \times m}$ Has At least one monomial column. Let the matrix (10) has $n_1 < n$ linearly independent monomial columns (12). The columns (13) are chosen so that the matrix (14) is monomial. It is assumed that the following condition

$$P_k^T A P_i = 0 \text{ for } i = 1,\dots,n_1;\ k = n_1 + 1,\dots,n \tag{27}$$

is satisfied. In a similar way as in case $m = 1$ the following theorem can be proved.

Theorem 5. Let the positive fractional system (1) be unreachable but the matrix (10) has n_1 linearly independent monomial columns (12) and the assumption (27) be satisfied. Then the pair (A_α, B) of the system can be reduced to the form (16) by the similarity transformation with the matrix (14). Moreover the positive pair $(\overline{A}_1, \overline{B}_1)$ is reachable and the pair $(\overline{A}_2, \overline{B}_2 = 0)$ is unreachable.

Example 2. Consider the fractional system (4) for $0 < \alpha < 1$ with the matrices

$$A = \begin{bmatrix} a_{11} - \alpha & 0 & 0 & 0 \\ 0 & 1-\alpha & 0 & 1 \\ 2 & a_{32} & 0 & 0 \\ 0 & 0 & 1 & -\alpha \end{bmatrix},\quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (a_{11} > 0,\ \ a_{32} > 0) \tag{28}$$

The matrix

$$A_\alpha = A + I_n \alpha = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & a_{32} & \alpha & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \Re_+^{4\times 4} \tag{29}$$

and the fractional system is positive. The pair (A_α, B) is unreachable since the matrix

$$[B \quad A_\alpha B] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \alpha \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{30}$$

has three linearly independent monomial columns. In this case the columns (12) are the first three columns of the matrix (30) and we choose $P_4 = [1 \quad 0 \quad 0 \quad 0]^T$. The matrix (14) has the form

$$P = [P_1 \quad P_2 \quad P_3 \quad P_4] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{31}$$

It is easy to check that the assumption (27) is satisfied since $P_4^T A_\alpha P_i = [a_{11} \quad 0 \quad 0 \quad 0] P_i = 0$ for $i = 1,2,3$. Using (28), (29) and (31) we obtain

$$\overline{A} = P^{-1} A_\alpha P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \alpha & a_{32} & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{11} \end{bmatrix} = \begin{bmatrix} \overline{A}_1 & \overline{A}_{12} \\ 0 & \overline{A}_2 \end{bmatrix},\quad \overline{B} = P^{-1} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix} \tag{32}$$

The pair $(\overline{A}_1, \overline{B}_1)$ where

$$\overline{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \alpha & a_{32} \\ 1 & 0 & 1 \end{bmatrix} \text{ and } \overline{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (33)$$

is reachable since the matrix

$$[\overline{B}_1 \quad \overline{\Phi}_1\overline{B}_1] = [\overline{B}_1 \quad \overline{A}_1\overline{B}_1] \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (34)$$

has three linearly independent monomial columns.
The considerations can be extended for the dual notion observability of the positive fractional linear discrete-time systems [4, 7].

4. CONCLUDING REMARKS

The idea of Kalman's decomposition theorem has been extended for positive fractional discrete-time linear systems. Conditions and procedure have been presented for decomposition of unreachable positive fractional discrete-time linear system into reachable and unreachable parts. The considerations have been illustrated by two numerical examples.
These considerations can be easily extended for 2D linear systems[3]. An open problem is an extension of this decomposition for positive standard and fractional continuous-time linear systems.

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5. REFERENCES

1. P.J. Antsaklis and A.N. Michel, *Linear Systems*, Birkhauser, Boston 2006.
2. L. Farina and S. Rinaldi, *Positive Linear Systems; Theory and Applications*, J. Wiley, New York, 2000.
3. T. Kaczorek, *Positive 1D and 2D systems*, Springer Verlag, London 2001.
4. T. Kaczorek, Decomposition of the pairs (*A*, *B*) and (*A*, *C*) of positive discrete-time linear systems, Archives of Control Sciences, vol. 20, no. 3, 2010, pp.253-273.
5. T. Kaczorek, Positive linear systems with different fractional orders, *Bul. Pol. Acad. Sci. Techn. Sci.*, vol. 58, no.3, 2010.
6. T. Kaczorek, Fractional positive continuous-time linear systems and their reachability, *Int. J. Appl. Math. Comput. Sci*. Vol. 18, No. 2, 2008, pp.223-228.
7. T. Kaczorek, *Selected problems in theory of fractional systems*, Publishing Co. of Bialystok University of Technology, Bialystok 2009 (in Polish).
8. T. Kaczorek, *Linear Control Systems*, Vol. 1, J. Wiley, New York 1993.
9. T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, New York 1980.
10. R.E. Kalman, Mathematical Descriptions of Linear Systems, *SIAM J. Control*, Vol. 1, 1963, pp.152-192.
11. R.E. Kalman, On the General Theory of Control Systems, *Proc. Of the First Intern. Congress on Automatic Control*, Butterworth, London, 1960, pp.481-493.
12. P. Ostalczyk, *Epitome of the Fractional Calculus: Theory and its Applications in Automatics*, Wydawnictwo Politechniki Łódzkiej, Łódź 2008 (in Polish).
13. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego 1999.

14. A.G. Radwan, A.M. Soliman, A.S. Elwakil and A. Sedeek, On the stability of linear systems with fractional-order elements, *Chaos, Solitons and Fractals*, vol. 40, no. 5, 2009, pp. 2317-2328.
15. H.H. Rosenbrock, *State-Space and Multivariable Systems Theory*, J. Wiley, New York 1974.
16. A. Ruszewski, Stability regions of closed-loop system with time delay inertial plant of fractional order and fractional order PI controller, *Bull. Pol. Acad. Sci. Tech.*, vol. 56, no. 4, 2008, pp.329-332.
17. S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Newark 1993.
18. X.J. Wen, Stability analysis of a class of nonlinear fractional-order systems, *IEEE Trans. Cir. and Syst.II*, vol. 55, Nov., 2008, pp.1178-1182.
19. W.A. Wolovich, *Linear Multivariable Systems*, Springer-Verlag New York 1974.
20. B.M. Vinagre, C.A. Monje and A.J. Calderon, Fractional order systems and fractional order control actions, *41th IEEE Conf. on Decision and Control*, Las Vegas NV, December 2002.