Observability of linear discrete-time systems with different fractional orders

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Abstract: In the paper the observability problem for the linear discrete-time positive systems with different fractional orders is presented. Necessary and sufficient conditions for observability of this class of dynamical systems are given. A method for computing the initial state is proposed. Considerations are illustrated by theoretical example. Numerical calculations have been performed in the MATLAB program environment.

Keywords: fractional, positive, discrete-time, system, observability

Observability is one of the fundamental concepts in the control theory. Systematic studies of observability were started at the beginning of sixties, when the theory of observability based on description in the form of the state space for linear control systems was worked out. This problem, first given by R.E. Kalman [1], plays a crucial role in the study of canonical forms of dynamical systems or observer synthesis [2]. Roughly speaking, observability denotes studying the possibility of estimating the initial state from the output signal. In this paper, the problem of observability of positive fractional systems will be considered. In positive dynamical systems each inputs, state variables and outputs take only non-negative values. Examples of such systems are processes involving chemical reactors, distillation column, compartmental systems or atmospheric pollution models [3]. Dynamical systems described by fractional order differential or difference equations have been investigated in several areas such as viscoelasticity, diffusion processes, electrochemistry, control theory, electrical engineering, etc. (see [4–10] and references therein, for example).

The conditions of observability of standard and positive discrete-time integer order systems we can found in [4, 11, 12]. The problem of observability of fractional systems has been considered in [10, 13–15], for example.

In this paper, the basic definition as well as conditions for the observability of the positive linear discrete-time systems with different fractional orders are formulated. These conditions are partially based on conditions given in [10, 16] for positive discrete-time integer order systems and for standard discrete-time fractional systems, respectively. Moreover, a method for computing the nonnegative initial state of the considered systems is proposed.

The considerations presented in this paper are the complement of the general control theory of the fractional discrete-time systems and can be applied in different areas of sciences.

1. Problem statement

In this paragraph some basic notations and definitions are introduced which will be used in the next parts of the paper.

Let $\Re^{n\times m}$ by the set of $n\times m$ real matrices and $\Re^{n\times 1}$. The set of $n\times m$ matrices with nonnegative entries will be denoted by $\Re_+^{n\times m}$ and $\Re_+^n = \Re_+^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n\times n$ identity matrix will be denoted by I_n.

In the theory of discrete-time systems with the same fractional order for each state equation the following definition of the fractional discrete derivative is used [10,12]:

$$\Delta^\alpha x_i = \sum_{k=0}^{i} (-1)^k \binom{\alpha}{k} x_{i-k} \tag{1}$$

where $\alpha \in \Re$ is the order of the fractional difference, $k \in Z_+$ is the number of samples for which the derivative is calculated and

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for} \quad k=0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} & \text{for} \quad k=1,2,... \end{cases} \tag{2}$$

The state equations of the system mentioned above has the form [10,12]:

$$\Delta^\alpha x_{i+1} = Ax_i + Bu_i, \quad i \in Z_+, \tag{3}$$

$$y_i = Cx_i + Du_i, \tag{4}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$ and $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors.

Without loss of generality we can take the input $u_i = 0$, $i \in Z_+$.

Let us consider the discrete-time system [10, 17]:

$$\Delta^{\overline{\alpha}} x_{i+1} = Ax_i + Bu_i, \quad i \in Z_+,$$
$$x_{i+1} = \Delta^{\overline{\alpha}} x_{i+1} - \sum_{j=1}^{k+1} (-1)^j \overline{\alpha} x_{i+1-j}, \tag{5}$$

$$y_i = Cx_i, \tag{6}$$

with different orders α ($\alpha_1 \neq \alpha_2 \neq \ldots \neq \alpha_n$) for each state equation where:

$$x_i = \begin{bmatrix} x_i^1 \\ \vdots \\ x_i^n \end{bmatrix}, \quad \Delta^{\overline{\alpha}} x_{i+1} = \begin{bmatrix} \Delta^{\alpha_1} x_i^1 \\ \vdots \\ \Delta^{\alpha_n} x_{i+1}^n \end{bmatrix}, \alpha_r \in (0,1), r = 1, \ldots, n$$
$$\overline{\alpha}_k = diag\left[\begin{pmatrix} \alpha_1 \\ k \end{pmatrix} \ \cdots \ \begin{pmatrix} \alpha_n \\ k \end{pmatrix}\right]. \tag{7}$$

The solution of (5) with the initial condition x_0 is given by the formula [10]:

$$x_k = \Phi_k^{\overline{\alpha}} x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1}^{\overline{\alpha}} B u_i; \quad x_0 = \begin{bmatrix} x_0^1 \\ \vdots \\ x_0^n \end{bmatrix} \in \Re^n, \tag{8}$$

where $\Phi_k^{\overline{\alpha}}$ is determined by the equation:

$$\Phi_{k+1}^{\overline{\alpha}} = (A + \overline{\alpha}_1)\Phi_k^{\overline{\alpha}} + \sum_{j=2}^{k+1} (-1)^{j+1} \overline{\alpha}_j \Phi_{k-i+1}^{\overline{\alpha}}, \tag{9}$$

with $\Phi_0^{\overline{\alpha}} = I_n$.

The following conditions for the system (5), (6) can be proved in the same manner as for the positive fractional system (3), (4), see [12], for example.

Lemma 1. If

$$0 < \alpha_l < 1, \quad l = 1, \ldots, n \tag{10}$$

then

$$(-1)^{i+1} \overline{\alpha}_k > 0. \tag{11}$$

where $\overline{\alpha}_k$ has the form (7).

Lemma 2. If (10) holds and

$$(A + \overline{\alpha}_1) \in \Re_+^{n \times n} \tag{12}$$

then

$$\Phi_k^{\overline{\alpha}} \in \Re_+^{n \times n}, \quad k = 1,2,\ldots \tag{13}$$

Theorem 1. The fractional system (5), (6) is positive (internally) if and only if

$$A_{\overline{\alpha}} = (A + \overline{\alpha}_1) \in \Re_+^{n \times n}, \ B \in \Re_+^{n \times m}, \ C \in \Re_+^{p \times n}. \tag{14}$$

Now we shall formulate the fundamental definitions for observability of the fractional positive system (5), (6).

Definition 1. The state $x_0 \in \Re_+^n$ of the system (5), (6) is called observable if knowledge of the output $y_i \in \Re_+^p$, $i = 0,1,\ldots,q-1$, suffices to determine uniquely x_0.

Definition 2. The system (5), (6) is called observable in q steps, if it is possible to find the initial conditions $x_0 \in \Re_+^n$ knowing the system output in q points (steps) $y_i \in \Re_+^p$, $i = 0,1,\ldots,q-1$, generated by this condition.

Definition 3. The system (5), (6) is called observable, if there exists a natural number $q \geq 1$ such that the system is observable in q steps.

The main purpose of the paper is to give necessary and sufficient conditions for observability of discrete-time positive fractional systems (5), (6) with different fractional orders in the state equations ($\alpha_1 \neq \alpha_2 \neq \ldots \neq \alpha_n$).

2. Solution of the problem

Taking into account that $u_i = 0$, $i \in Z_+$ and substituting the solution of the equation (5) of the form (8) into the output equation (6) we obtain:

$$y_i = Cx_i = C\Phi_k^{\overline{\alpha}} x_0 \in \Re_+^{qp}, \quad x_0 = \begin{bmatrix} x_0^1 \\ \vdots \\ x_0^n \end{bmatrix} \in \Re_+^n. \tag{15}$$

From (15) for $k = 0,1,\ldots,q-1$ we obtain:

$$\begin{aligned} y_0 &= C\Phi_0^{\overline{\alpha}} = Cx_0, \\ y_1 &= C\Phi_1^{\overline{\alpha}}, \\ &\vdots \\ y_{q-1} &= C\Phi_{q-1}^{\overline{\alpha}}. \end{aligned} \tag{16}$$

The above equations we can write in the form:

$$y_0^q = S_q x_0 \tag{17}$$

where

$$y_0^q = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix} \in \Re_+^{qp}, \ S_q = \begin{bmatrix} C \\ C\Phi_1^{\overline{\alpha}} \\ \vdots \\ C\Phi_{q-1}^{\overline{\alpha}} \end{bmatrix} \in \Re_+^{qp \times n}, \tag{18}$$

and S_q is called the observability matrix.

Let e_j, $j = 1,\ldots,n$ be the jth row of the identity matrix I_n. A row ae_j for $a > 0$ is called a monomial row.

Now we shall formulate the criteria for observability of the fractional positive system (5), (6).

Theorem 2. The system (5), (6) is observable in q steps if and only if the observability matrix S_q (18) contains n linearly independent monomial rows.

Proof. From Definition 2 and (18) it follows that for $y_i \in \Re_+^p$, $i = 0,1,\ldots,q-1$, there exists $x_0 \in \Re_+^n$ if and only if the matrix (18) contains n linearly independent monomial rows.

Theorem 3. If the matrix

$$\overline{S} = \begin{bmatrix} C \\ A_{\overline{\alpha}} \end{bmatrix} \in \Re_+^{(p+n)\times n}, \tag{19}$$

has not n linearly independent monomial rows, then does not exists $q \in Z_+$ such that the system (5), (6) is observable.

Proof. If the system (5), (6) is observable then the matrix (18) has n linearly independent monomial rows. It is possible only if the matrix (19) has n linearly independent monomial rows. Moreover the number of monomial rows in (18) cannot be greater than in (19).

From the above considerations it follows that if the fractional positive system (5), (6) is observable then it is always observable in q steps and if the systems is observable in q steps then it is always observable in greater number of steps ($q + t$, $t = 1,2,...$). The number $q \in Z_+$ we can calculate using the following simple condition:

Theorem 4. If the system (5), (6) is observable then it is observable in q steps with $q \ge E[n/l]$ where $E[n/l]$ denotes the minimal integer number grater or equal to n/l and l is the number of linearly independent rows of the matrix C.

Proof. Each matrix $C\Phi_k^{\overline{\alpha}}$, $k = 0,1,...,q-1$ of the observability matrix (18) may has maximum l linearly independent monomial rows. Hence, if the fractional positive system (5), (6) is observable then $ql = n$.

The initial state $x_0 \in \Re_+^n$ of the system (5), (6) we can calculate using the following condition:

Theorem 5. If the matrix S_q (18) contains q linearly independent monomial rows and $[S_q^T S_q]^{-1} S_q^T \in \Re_+^{n\times qp}$ then knowing the output q_0^q the initial state x_0 of the positive fractional system (5), (6) can be computed from:

$$x_0 = [S_q^T S_q]^{-1} S_q^T y_0^q \in \Re_+^n \tag{20}$$

Proof. If there exists $q \in Z_+$ such that the matrix (18) has q linearly independent monomial rows ($\mathrm{rank} S_q = n$) then $\det[S_q^T S_q] \ne 0$ and the matrix $[S_q^T S_q]^{-1}$ is well defined. If $[S_q^T S_q]^{-1} S_q^T \in \Re_+^{n\times qp}$ then knowing the output $y_0^q \in \Re_+^{qp}$ we get $x_0 \in \Re_+^n$.

The formula (20) is based on the left-inverse of the observability matrix (18). Using another form of left-inverse of the rectangular matrix (see [6], for example) we can write the following formula:

$$x_0 = \{[KS_q]^{-1} K\} y_0^q \in \Re_+^n \tag{21}$$

where $K \in \Re_+^{n\times qp}$ we can choose arbitrarily, but such that $\det(KS_q) \ne 0$. Let us notice that in the special case if we choose $K = S_q^T$ then from (21) we get the formula (20).

From the above considerations it follows that the system (5), (6) is observable if and only if there exists a natural number $q \in Z_+$ which satisfies conditions given in Theorem 2 and 3.

3. Example

Let us consider the fractional system described by the equations (5), (6) with

$$A = \begin{bmatrix} -0.38 & 0.45 \\ 0 & -0.15 \end{bmatrix}, \quad \alpha_1 = 0.4, \quad \alpha_2 = 0.7, \tag{22}$$

and the output matrices

$$C_{(1)} = [0.55 \quad 0], \tag{23}$$

$$C_{(2)} = \begin{bmatrix} 0.55 & 0 \\ 0 & 1 \end{bmatrix}. \tag{24}$$

Test the observability of the system: 1) with (22), (23) and 2) with (22), (24).

We will check the positivity of the system. Using (12) we obtain:

$$A_{\overline{\alpha}} = A + \overline{\alpha}_1 = A + diag\left[\begin{pmatrix} \alpha_1 \\ 1 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ 1 \end{pmatrix}\right] = \\ = \begin{bmatrix} 0.02 & 0.45 \\ 0 & 0.55 \end{bmatrix} \in \Re_+^{2\times 2}. \tag{25}$$

Thus, by Theorem 1 the considered fractional system is positive. From Theorem 4 it follows that the system can be observable in $q \ge 2$ steps. Computing the matrix (18) for $q = 3$ with (22) and (23) we get:

$$S_q = S_3 = \begin{bmatrix} C_{(1)} \\ C_{(1)}\Phi_1^{\overline{\alpha}} \\ C_{(1)}\Phi_2^{\overline{\alpha}} \end{bmatrix} = \begin{bmatrix} 0.55 & 0 \\ 0.01 & 0.25 \\ 0.07 & 0.14 \end{bmatrix} \in \Re_+^{3\times 2}, \tag{26}$$

where

$$\Phi_1^{\overline{\alpha}} = (A + \overline{\alpha}_1)\Phi_0^{\overline{\alpha}} = A_{\overline{\alpha}}\Phi_0^{\overline{\alpha}} = \begin{bmatrix} 0.02 & 0.45 \\ 0 & 0.55 \end{bmatrix} \in \Re_+^{2\times 2},$$

$$\Phi_2^{\overline{\alpha}} = A_{\overline{\alpha}}\Phi_0^{\overline{\alpha}} - \overline{\alpha}_2 = \Phi_1^{\overline{\alpha}} - \overline{\alpha}_2 = \begin{bmatrix} 0.12 & 0.26 \\ 0 & 0.41 \end{bmatrix} \in \Re_+^{2\times 2}. \tag{27}$$

The observability matrix (26) does not contain $n = 2$ linearly independent monomial rows. Thus, by Theorem 2 the fractional system with (22) and (23) is not observable. Let us notice that the sufficient condition given in Theorem 3 is satisfied. In this case we cannot guarantee that the initial condition x_0 will be computed correctly.

Repeating calculations with (22) and the matrix (24) for $q = 2$ we obtain:

$$S_q = S_2 = \begin{bmatrix} C_{(2)} \\ C_{(2)}\Phi_1^{\overline{\alpha}} \end{bmatrix} = \begin{bmatrix} 0.25 & 0 \\ 0 & 1.00 \\ 0.01 & 0.11 \\ 0 & 0.55 \end{bmatrix} \in \Re_+^{4\times 2}. \tag{28}$$

Thus, by Theorem 2 the considered system with (22), (24) is observable in $q = 2$ steps. The first one and the last one row of the observability matrix (28) are monomial and linearly independent. Let us notice that observability of the system in this case depends form the monomial structure of the output matrix $C_{(2)}$.

Let us assume that the output sequence q_0^q of the system with (22), (24) logged in the first $q = 2$ measuring points has the following form:

$$y_0^2 = \begin{bmatrix} y_0^0 & y_0^1 \end{bmatrix}^T \in \Re_+^4; \quad y_0^0 = \begin{bmatrix} 0.63 \\ 1.50 \end{bmatrix}, y_0^1 = \begin{bmatrix} 0.18 \\ 0.82 \end{bmatrix}. \tag{29}$$

Knowing the nonnegative output sequence (29) we can compute the initial state of the considered system. First we use the formula (20) and next the formula (21).
Substituting (29) and (28) into (20) we obtain the initial condition of the system in the form:

$$x_0 = [S_2^T S_2]^{-1} S_2^T y_0^2 = \begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1.5 \end{bmatrix} \in \Re_+^2. \tag{30}$$

Let the matrix K has the following structure:

$$K = \begin{bmatrix} 5 & 0 & 3 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix} \in \Re_+^{2\times 4}, \quad \det(KS_2) = 2.31. \tag{31}$$

Using (21) with (29) and (28) we obtain:

$$x_0 = \left\{ [KS_2]^{-1} K \right\} y_0^2 = \begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1.5 \end{bmatrix} \in \Re_+^2. \tag{32}$$

In this case, the nonnegative initial state vector of the considered fractional system also was computed correctly.

All calculations in the above example were performed using MATLAB package. The matrix K with nonnegative values we may also get applying *m*-function *rand*.

4. Concluding remarks

In the paper the positive linear discrete-time systems with different fractional order have been considered. The necessary and sufficient conditions for observability have been established by generalization of the results given in [11,17]. An example of the simple method (based on the knowledge of the output signal and the structure of the observability matrix (18) of the considered system) for computations of the nonnegative unknown initial state of the fractional system has been given.

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References

1. Kalman R.E., *On the general theory of control system.*, Proc. Of the 1st IFAC Congr., Butterworth, London 1960.
2. Kaczorek T., *Control theory and systems*, PWN, Warsaw 1996.
3. Kaczorek T., *Positive 1D and 2D systems*, Springer-verlag, London 2002.
4. Debnath L., *Recent applications of fractional calculus to science and engineering*, “Int. Journal of Mathematics and Mathematical Sciences”, Vol. 54, 2003, 3413–3442 [on-line: www.ijmms.hindawi.com].
5. Dzieliński A., Sierociuk D., Sarwas G., *Some applications of fractional order calculus*, “Bull. of the Polish Acad. Of Sciences, Technical Sciences”, Vol. 58, No. 4, 2010, 583–592.
6. Kilbas A.A., Srivastava H. M., Trujillo J. J., *Theory and applications of fractional differential equations.* Elsevier, Amsterdam 2006.
7. Lino P., Maione G., *Loop-shaping annd easy tuning of fractional-order proportional integral controllers for position servo systems*, “Asian Journal of Control”, 2012.
8. Podlubny I., *Fractional differential equations*, Acad. Press, San Diego 1999.
9. Sabatier J., Agraval O. P., Machado, *Advances in fractional calculus. Theoretical developments and applications in physics and engineering*, Springer, London 2007.
10. Sierociuk D., *Estimation and control of discrete-time dybnamical fractional systems described in state space,* Ph.D. thesis. Warsaw University of Technology, Warsaw 2007.
11. Kaczorek T., *Selected problems of fractional systems theory,* Springer, Berlin 2011.
12. Kaczorek T., *Vectors and matrices in automatics and electrotechnics*, WNT, Warsaw 1998.
13. Bettayeb M., Djennoune S., *A note on the controllability and the observability of fractional dynamical systems*, [in:] Proc. of the 2nd IFAC Workshop on Fractional Differentation and its Applications, Porto, Portugal 2006, 506–511.
14. Mantignon D., d’Andrea-Novel B., *Some results on controllability and observability of finite-dimensional fractional differential systems*, [in:] Proc. of the IMACS. IEEE SMC Conf., France 1996, 952–956.
15. Mozyrska D., Pawłuszewicz E., *Observability of linear q-difference fractional order systems with finite initial memory*, “Bull. of the Polish Acad. of Sci.”, Vol. 58, No. 4, 2010, 601–605.
16. Kociszewski R., *Controllability and observability of linear time-invariant positive discrete-time systems with delays*, Ph.D. thesis, Białystok University of Technology, Białystok 2008.
17. Busłowicz M., *Simple analytic conditions for stability of fractional discrete-time linear systems with diagonal state matrix*, “Bull. of the Polish Acad. of Sciences, Technical Sciences” (in press).

Obserwowalność liniowych układów dyskretnych różnych niecałkowitych rzędów

Streszczenie: W pracy rozpatrzono problem obserwowalności układów dyskretnych dodatnich przy różnych niecałkowitych rzędach w równaniu stanu. Podano warunki konieczne i wystarczające obserwowalności rozpatrywanej klasy układów dynamicznych. Zaproponowano prostą metodę wyznaczania nieujemnego stanu początkowego takiego układu. Rozważania zilustrowano przykładem teoretycznym, zaś niezbędne obliczenia wykonano w środowisku programowym MATLAB.

Słowa kluczowe: rząd niecałkowity, układ, dodatni, dyskretny, obserwowalność

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