dr inż. Łukasz Sajewski
Bialystok University of Technology
Faculty of Electrical Engineering

DODATNIE LINIOWE UKŁADY CIĄGŁO-DYSKRETNE Z OPÓŹNIENIEM W WEKTORZE STANU

Zaprezentowana zostanie nowa klasa dodatnich układów ciągło-dyskretnych z opóźnieniem w wektorze stanu. Rozpatrzone zostaną trzy modele opisujące tę klasę układów w przestrzeni stanu. Podane zostaną warunki konieczne i wystarczające dodatniości liniowych układów ciągło-dyskretnych z opóźnieniem w wektorze stanu. Podana zostanie postać operatorowej macierzy transmitancji właściwych rozpatrywanej klasy układów.

POSITIVE CONTINUOUS-DISCRETE TIME LINEAR SYSTEMS WITH DELAYS IN STATE VECTOR

A new class of positive continuous-discrete time linear systems with delays in state vector is addressed. Three state space model of this class of linear systems are considered. Necessary and sufficient conditions for the positivity of continuous-discrete time systems with delays in state vector are established. The proper transfer matrix of this class of linear systems is given.

1. INTRODUCTION

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems is given in the monographs [6, 9].

2D hybrid system is dynamic systems that incorporate both continuous-time and discrete-time dynamics. It means that state vector, input and output vectors of 2D hybrid system depend on the continuous time t and the discrete variable i. Examples of hybrid systems include systems with relays, switches and hysteresis, transmissions, and other motion controllers, constrained robotic systems, automated highway systems, flight control, management systems and analog/digital circuit. The positive continuous-discrete 2D linear systems have been introduced in [8], positive hybrid linear systems in [10] and the positive fractional 2D hybrid systems in [11]. Different methods of solvability of 2D hybrid linear systems have been discussed in [13] and the solution to singular 2D hybrids linear systems has been derived in [15]. The realization problem for positive 2D hybrid systems has been addressed in [12]. Some problems of dynamics and control of 2D hybrid systems have been considered in [5, 7]. The problems of stability and robust stability of 2D continuous-discrete linear systems have been investigated in [1-4]. Recently the stability and robust stability of Fornasini-Marchesini type model and of Roesser type model of scalar continuous-discrete linear systems have been analyzed by Buslowicz in [2–4].

The main goal of this paper is to present a new class of positive continuous-discrete time linear systems with delays in state vector. Necessary and sufficient conditions for positivity of this class of linear systems will be established. Three state space model called: general model, first Fornasini-Marchesini model and second Fornasini-Marchesini model of linear continuous-discrete time systems with delays will be considered. The proper transfer matrix of this class of linear systems will be given.

The paper is organized as follows. In subsection 2.1 the general model of the continuous-discrete time linear systems with delays and its boundary conditions are presented. In this subsection the definition and conditions for positivity of this class of linear systems are given. Subsections 2.2 and 2.3 contains the same considerations as in 2.1 for first Fornasini-Marchesini model and second Fornasini-Marchesini model respectively. In subsection 2.4 results according to transfer matrix of the continuous-discrete time systems with delays are presented. Concluding remarks are given in section 3.

The following notation will be used: $\Re$ - the set of real numbers, Z_+ – the set of nonnegative integers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. THREE MODELS OF POSITIVE CONTINUOUS-DISCRETE TIME SYSTEMS WITH DELAYS

2.1 General model of the continuous-discrete time system with delays

Consider a hybrid system with delays described by the equations

$$\begin{aligned}\dot{x}(t,i+1) = {} & A_0x(t,i) + A_1\dot{x}(t,i) + A_2x(t,i+1) + A_3x(t-d,i+1) + A_4\dot{x}(t,i-1) + A_5x(t-d,i-1) \\ & + B_0u(t,i) + B_1\dot{u}(t,i) + B_2u(t,i+1)\end{aligned} \tag{2.1a}$$

$$y(t,i) = Cx(t,i) + Du(t,i),\ t\in\Re_+ = [0,+\infty],\ i\in Z_+ = \{0,1,...\} \tag{2.1b}$$

where $\dot{x}(t,i) = \dfrac{\partial x(t,i)}{\partial t}$, $x(t,i)\in\Re^n$, $u(t,i)\in\Re^m$, $y(t,i)\in\Re^p$ and $A_k\in\Re^{n\times n}$, $k = 0,1,2,3,4$; $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$ are the real matrices, $d>0$ is a delay.

Boundary conditions for (1a) have the form

$$x_{0i}(t,i),\ t\in[-d,0],\ i\in Z_+,\ x_{t0}(t,i),\ \dot{x}_{t0}(t,i),\ i\in[-1,0],\ t\in\Re_+ \tag{2.2a}$$

and

$$x_{0i}(t,0) = 0,\ t\in[-d,0],\ x_{t0}(0,i) = \dot{x}_{t0}(0,i) = 0,\ i\in[-1,0] \tag{2.2b}$$

Definition 2.1. The hybrid system with delays (2.1) is called (internally) positive if $x(t,i)\in\Re_+^n$ and $y(t,i)\in\Re_+^p$, $t\in\Re_+$, $i\in Z_+$ for arbitrary boundary conditions $x_{0i}(t,i)\in\Re_+^n$, $t\in[-d,0]$, $i\in Z_+$ i $x_{t0}(t,i)\in\Re_+^n$, $\dot{x}_{t0}(t,i)\in\Re_+^n$, $i\in[-1,0]$, $t\in\Re_+$ and all inputs $u(t,i)\in\Re_+^m$, $t\in\Re_+$, $i\in Z_+$.

Theorem 2.1. The hybrid system with delays (2.1) is internally positive if and only if

$$\begin{gathered}A_2\in M_n,\ \ A_0,A_1,A_3,A_4,A_5\in\Re_+^{n\times n},\ \ A_0 + A_1A_2\in\Re_+^{n\times n} \\ B_0,B_1,B_2\in\Re_+^{n\times m},\ \ C\in\Re_+^{p\times n},\ \ D\in\Re_+^{p\times m}\end{gathered} \tag{2.3}$$

Proof.

Necessity. Let e_i be the i-th ($i = 1,...,n$) column of the identity matrix I_n. From (2.1) for $t\in\Re_+$, $i = 0$ and $x(t,1) = e_i$, $x(t,0) = 0$, $\dot{x}(t,0) = 0$, $x(t-d,1) = 0$, $\dot{x}(t,-1) = 0$, $x(t-d,-1) = 0$ and inputs $u(t,0) = \dot{u}(t,0) = u(t,1) = 0$ we have $\dot{x}(t,1) = A_2e_i$. The trajectory does not live the orthant R_+^n only if $A_2e_i\geq 0$, what implies $a_{ij}\geq 0$, $i\neq j$. Therefore, the matrix A_2 has to be the Metzler matrix. For the same reasons for $x(0,1) = 0$, $x(0,0) = 0$, $\dot{x}(0,0) = 0$, $x(-d,1) = 0$, $\dot{x}(0,-1) = 0$, $x(t-d,-1) = 0$, $\dot{x}(0,1) = B_0u(0,0) + B_1\dot{u}(0,0) + B_2u(0,1)\in\Re_+^n$, what implies $B_0 + B_1 + B_2\in R_+^{n\times m}$ since

$u(0,0)$, $\dot{u}(0,0)$, $u(0,1) \in R_+^m$ may be arbitrary. Similarly, for $t \in R_+$, $i = 0$ and $x(t,1) = 0$, $x(t,0) = 0$, $x(t-d,1) = 0$, $\dot{x}(t,-1) = 0$, $x(t-d,-1) = 0$ we have $\dot{x}(t,1) = A_1\dot{x}(t,0) \in \Re_+^n$ what implies $A_1 \in \Re_+^{n\times n}$ since $\dot{x}(t,0) \in \Re_+^n$ may be arbitrary. For the same reasons for $x(0,1) = 0$, $\dot{x}(0,0) = 0$, $x(-d,1) = 0$, $\dot{x}(0,-1) = 0$, $x(t-d,-1) = 0$, $\dot{x}(0,1) = A_0 x(0,0) \in \Re_+^n$, what implies $A_0 \in \Re_+^{n\times n}$ since $x(0,0) \in \Re_+^n$ may be arbitrary. From (1b) for $u(0,0) = 0$ we have $y(0,0) = Cx(0,0) \in \Re_+^p$, what implies $C \in \Re_+^{p\times n}$, since $x(0,0) \in \Re_+^n$ may be arbitrary. For the same reasons for $x(0,0) = 0$ we have $y(0,0) = Du(0,0) \in \Re_+^p$, what implies $D \in \Re_+^{p\times m}$, since $u(0,0) \in \Re_+^m$ may be arbitrary.

In a similar way we may show that the hybrid system with delays (2.1) is internally positive only if the conditions (2.2) are satisfied.

Sufficiency. From (2.1) for $i = 0$ and $t \in [0,d]$ we have

$$\dot{x}(t,1) = A_2 x(t,1) + A_3 x(t-d,1) + F(t,0) \tag{2.4a}$$

where

$$F(t,0) = A_0 x(t,0) + A_1\dot{x}(t,0) + A_4\dot{x}(t,-1) + A_5 x(t-d,-1) + B_0 u(t,0) + B_1\dot{u}(t,0) + B_2 u(t,1) \tag{2.4b}$$

For given nonnegative initial conditions

$x(t,0)$, $\dot{x}(t,0)$, $\dot{x}(t,-1)$, $x(t-d,-1)$, $x(t-d,1) \approx x_{0i}(0,1)$ and $u(t,0), \dot{u}(t,0), u(t,1) \in \Re_+^m$, $t \in [0,d]$, we obtain $F(t,0) \in \Re_+^n$, $t \in [0,d]$ if $A_0, A_1, A_4, A_5 \in \Re_+^{n\times n}$, $B_0, B_1, B_2 \in \Re_+^{n\times m}$. he solution of the equation (2.4a) has the form

$$x(t,1) = e^{A_2 t}x(0,1) + \int_0^t e^{A_2(t-\tau)}\big(A_3 x(\tau-d,1) + F(\tau,0)\big)d\tau \tag{2.5}$$

and is nonnegative since $e^{A_2 t} \in \Re_+^{n\times n}$ for $t \in [0,d]$ if and only if A_2 is the Metzler matrix and $A_3 \in \Re_+^{n\times n}$. Knowing $x(t,1)$ for $t \in [0,d]$ in a similar way we can find $x(t,1)$ for $t \in [d,2d]$, $t \in [2d,3d], \ldots$.

From (2.1a) for $i = 1$ we have

$$\dot{x}(t,2) = A_2 x(t,2) + A_3 x(t-d,2) + F(t,1) \tag{2.6a}$$

where

$$F(t,1) = A_0 x(t,1) + A_1\dot{x}(t,1) + A_4\dot{x}(t,0) + A_5 x(t-d,0) + B_0 u(t,1) + B_1\dot{u}(t,1) + B_2 u(t,2) \tag{2.6b}$$

Substituting (2.5) into (2.6b) we obtain

$$\begin{aligned}
F(t,1) &= A_0 e^{A_2 t}x(0,1) + A_0\int_0^t e^{A_2(t-\tau)}A_3 x(\tau-d,1)d\tau + A_0\int_0^t e^{A_2(t-\tau)}F(\tau,0)d\tau \\
&\quad + A_1\frac{d}{dt}\left(e^{A_2 t}x(0,1) + \int_0^t e^{A_2(t-\tau)}A_3 x(\tau-d,1)d\tau + \int_0^t e^{A_2(t-\tau)}F(\tau,0)d\tau\right) \\
&\quad + A_4\dot{x}(t,0) + A_5(t-d,0) + B_0 u(t,1) + B_1\dot{u}(t,1) + B_2 u(t,2) \qquad (2.6c)\\
&= e^{A_2 t}(A_0 + A_1 A_2)\big(x(0,1) - F(0,0) - A_3 x(-d,1)\big) + A_0\big(A_3 x(t-d,1) + F(t,0)\big) \\
&\quad + A_1\big(A_3\dot{x}(t-d,1) + \dot{F}(t,0)\big) + A_4\dot{x}(t,0) + A_5(t-d,0) + B_0 u(t,1) + B_1\dot{u}(t,1) + B_2 u(t,2)
\end{aligned}$$

For given nonnegative initial conditions (2.2), nonnegative inputs we obtain $F(t,1) \in \Re_+^n$, f i and only if $A_0, A_1, A_4, A_5 \in \Re_+^{n\times n}$, $A_0 + A_1 A_2 \in \Re_+^{n\times n}$, $B_0, B_1, B_2 \in \Re_+^{n\times m}$. The solution of the equation (2.6a) has the form

$$x(t,2)=e^{A_2 t}x(0,2)+\int_0^t e^{A_2(t-\tau)}\big(A_3 x(\tau-d,2)+F(\tau,1)\big)d\tau\in\Re_+^n \tag{2.7}$$

if the conditions (2.3) are met.
Continuing the procedure for $i>0$ with nonnegative initial conditions (2.2), nonnegative inputs and conditions (3) we may show that

$$F(t,i)=A_0x(t,i)+A_1\dot{x}(t,i)+A_4\dot{x}(t,i-1)+A_5x(t-d,i-1)+B_0u(t,i)+B_1\dot{u}(t,i)+B_2u(t,i+1)\in\Re_+^n \tag{2.8a}$$

and

$$x(t,i+1)=e^{A_2 t}x(0,i+1)+\int_0^t e^{A_2(t-\tau)}\big(A_3x(\tau-d,i+1)+F(\tau,i)\big)d\tau\in\Re_+^n \tag{2.8b}$$

for $t\in\Re_+$ and $i\in Z_+$. □

2.2. First Fornasini-Marchesini model of continuous-discrete time system with delays

Let consider a general model (2.1) with $B=B_0$, $B_1=B_2=0$. We obtain so called first Fornaisni-Marchesini model of the continuous-discrete time system with delays in state vector described by the equations

$$\begin{aligned}\dot{x}(t,i+1)&=A_0x(t,i)+A_1\dot{x}(t,i)+A_2x(t,i+1)\\&\quad+A_3x(t-d,i+1)+A_4\dot{x}(t,i-1)+A_5x(t-d,i-1)+Bu(t,i)\end{aligned} \tag{2.9a}$$

$$y(t,i)=Cx(t,i)+Du(t,i),\ t\in\Re_+=[0,+\infty],\ i\in Z_+=\{0,1,...\} \tag{2.9b}$$

where $\dot{x}(t,i)=\dfrac{\partial x(t,i)}{\partial t}$, $x(t,i)\in\Re^n$, $u(t,i)\in\Re^m$, $y(t,i)\in\Re^p$ and $A_k\in\Re^{n\times n}$, $k=0,1,2,3,4,5;$ $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$ are the real matrices, $d>0$ is a delay.
Boundary conditions for (9a) have the form (2.2). The necessary and sufficient conditions for the positivity of this model is given by the following theorem.

Theorem 2.2. The continuous-discrete time system with delays (2.9) is internally positive if and only if

$$\begin{aligned}&A_2\in M_n,\quad A_0,A_1,A_3,A_4\in\Re_+^{n\times n},\quad A_0+A_1A_2\in\Re_+^{n\times n}\\&B\in\Re_+^{n\times m},\quad C\in\Re_+^{p\times n},\quad D\in\Re_+^{p\times m}\end{aligned} \tag{2.10}$$

The proof is similar as in Theorem 2.1.

2.3. Second Fornasini-Marchesini model of continuous-discrete time system with delays

Let consider a general model (2.1) with $A_0=0$ and $B_0=0$. We obtain so called second Fornaisni-Marchesini model of the continuous-discrete time system with delays in state vector described by the equations

$$\dot{x}(t,i+1)=A_1\dot{x}(t,i)+A_2x(t,i+1)+A_3x(t-d,i+1)+A_4\dot{x}(t,i-1)+B_1\dot{u}(t,i)+B_2u(t,i+1) \tag{2.11a}$$

$$y(t,i)=Cx(t,i)+Du(t,i),\ t\in\Re_+=[0,+\infty],\ i\in Z_+=\{0,1,...\} \tag{2.11b}$$

where $\dot{x}(t,i)=\dfrac{\partial x(t,i)}{\partial t}$, $x(t,i)\in\Re^n$, $u(t,i)\in\Re^m$, $y(t,i)\in\Re^p$ and $A_k\in\Re^{n\times n}$, $k=1,2,3,4;$ $B_1,B_2\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$ are the real matrices, $d>0$ is a delay.
Boundary conditions for (1a) have the form (2.2). The necessary and sufficient conditions for the positivity of this model is given by the following theorem.

Theorem 2.3. The hybrid system with delays (2.1) is internally positive if and only if

$$\begin{aligned}&A_2\in M_n,\quad A_1,A_3,A_4\in\Re_+^{n\times n},\quad A_1A_2\in\Re_+^{n\times n}\\&B_1,B_2\in\Re_+^{n\times m},\quad C\in\Re_+^{p\times n},\quad D\in\Re_+^{p\times m}\end{aligned} \tag{2.12}$$

The proof is similar as in Theorem 2.1.

2.4. Transfer matrix of the continuous-discrete time system with delays

Let consider the general model of the continuous-discrete time system with delays in state vector (2.1). Using the Laplace transform ($\mathcal{L}$) with zero boundary conditions according to t we obtain

$$
\begin{aligned}
sX(s,i+1) &= A_0X(s,i) + A_1sX(s,i) + A_2X(s,i+1) + A_3e^{-sd}X(s,i+1) \\
&\quad + A_4sX(s,i-1) + A_5e^{-sd}X(s,i-1) + B_0U(s,i) + B_1sU(s,i) + B_2U(s,i+1) \qquad (2.13a) \\
Y(s,i) &= CX(s,i) + DU(s,i)
\end{aligned}
$$

Then using Z ($\mathcal{Z}$) transform with zero boundary conditions according to i we obtain

$$
\begin{aligned}
szX(s,z) &= A_0X(s,z) + A_1sX(s,z) + A_2zX(s,z) + A_3e^{-sd}zX(s,z) + A_4z^{-1}sX(s,z) \\
&\quad + A_5e^{-sd}z^{-1}X(s,z) + B_0U(s,i) + B_0U(s,z) + B_1sU(s,z) + B_2zU(s,z) \qquad (2.13b) \\
Y(s,z) &= CX(s,z) + DU(s,z)
\end{aligned}
$$

Substituting $w_d = z^{-1}$, $w_c = e^{-sd}$ we have

$$
\begin{aligned}
X(s,z) &= C[I_nsz - (A_0 + A_5w_cw_d) - (A_1 + A_4w_d)s - (A_2 + A_3w_c)z]^{-1}(B_0 + B_1s + B_2z)U(s,z) \\
Y(s,z) &= CX(s,z) + DU(s,z)
\end{aligned} \qquad (2.12)
$$

where $X_k = X_k(s,z) = \mathcal{Z}\{\mathcal{L}[x_k(t,i)]\}$, $k = 1,2$; $U = U(s,z) = \mathcal{Z}\{\mathcal{L}[u(t,i)]\}$, $Y = Y(s,z) = \mathcal{Z}\{\mathcal{L}[y(t,i)]\}$.

From (2.12) we have the following theorem.

Theorem 2.4. Transfer matrix of the system (2.1) is given by equation

$$
\begin{aligned}
T(s,z) &= \frac{Y(s,z)}{U(s,z)} = T_{sp}(s,z) + D \\
&= C[I_nsz - (A_0 + A_5w_cw_d) - (A_1 + A_4w_d)s - (A_2 + A_3w_c)z]^{-1}(B_0 + B_1s + B_2z) + D
\end{aligned} \qquad (2.13)
$$

where $T_{sp}(s,z)$ is the strictly proper transfer matrix, $w_c = e^{-sd}$, $w_d = z^{-1}$ is the representation of delay for continuous-time and discrete-time part of the system (2.1) respectively.

Substituting $B = B_0$, $B_1 = B_2 = 0$ or $A_0 = 0$, $B_0 = 0$ into (2.13) we obtain transfer matrix of the first Fornasini-Marchesini model and second Fornasini-Marchesini model of the continuous-discrete time system with delays respectively.

For m input and p output continuous-discrete time system with delays in state vector the proper transfer matrix takes the form

$$
T(s,z) = \begin{bmatrix} T_{11}(s,z) & \dots & T_{1,m}(s,z) \\ \vdots & \dots & \vdots \\ T_{p,1}(s,z) & \dots & T_{p,m}(s,z) \end{bmatrix} \in \Re^{p\times m}(s,z) \qquad (2.14)
$$

where

$$
T_{k,l}(s,z) = \frac{\bar{b}^{k,l}_{n,n}s^nz^n + \bar{b}^{k,l}_{n,n-1}(w)s^nz^{n-1} + \bar{b}^{k,l}_{n-1,n}(w)s^{n-1}z^n + \ldots + \bar{b}^{k,l}_{10}(w)s + \bar{b}^{k,l}_{01}(w)z + \bar{b}^{k,l}_{00}(w)}{s^nz^n + \bar{a}^{k,l}_{n,n-1}(w)s^nz^{n-1} + \bar{a}^{k,l}_{n-1,n}(w)s^{n-1}z^n + \ldots + \bar{a}^{k,l}_{10}(w)s + \bar{a}^{k,l}_{01}(w)z + \bar{a}^{k,l}_{00}(w)} \qquad (2.15)
$$

for $k = 1,2,\ldots,p$; $l = 1,2,\ldots,m$; and

$$
\begin{aligned}
b^{k,l}_{p,q}(w) &= \sum_{i=0}^{n-p}\sum_{j=0}^{n-q} b^{pq,kl}_{i,j} w_c^{\,i} w_d^{\,j} \\
a^{k,l}_{p,q}(w) &= \sum_{i=0}^{n-p}\sum_{j=0}^{n-q} a^{pq,kl}_{i,j} w_c^{\,i} w_d^{\,j}
\end{aligned} \qquad (2.16)
$$

for $p,q = 0,1,...,n$, $\bar{b}^{k,l}_{n,n}(w) = \bar{b}^{k,l}_{n,n}$, $\bar{a}^{k,l}_{n_1,n_2}(w) = 1$ and $\Re^{p\times m}(s,z)$ is the set of $m \times n$ rational matrices in s and z.

Definition 2.2. The matrices (2.3) are called the positive realization of the transfer matrix $T(s,z)$ if they satisfy the equality (2.13).

3. CONCLUDING REMARKS

The new class of positive continuous-discrete time linear systems with delays in state vector has been introduced. Necessary and sufficient conditions for the positivity of this class of linear systems with delays has been established. Three model of continuous-discrete time linear systems with delays in state vector have been proposed. The proper transfer matrix of this class of linear system s has been given. The considerations can be also extended for fractional positive 2D continuous-discrete linear systems.

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