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PRAKTYCZNA STABILNOŚĆ ORAZ ASYMPTOTYCZNA STABILNOŚĆ STOŻKOWYCH UŁAMKOWYCH UKŁADÓW LINIOWYCH DYSKRETNYCH

Podano nową koncepcję praktycznej stabilności oraz asymptotycznej stabilności stożkowych liniowych ułamkowych układów dyskretnych. Sformułowano i udowodniono warunki konieczne i wystarczające dla praktycznej stabilności oraz asymptotycznej stabilności stożkowych układów ułamkowych. Wykazano, że: 1) stożkowe układy ułamkowe są praktycznie stabilne wtedy i tylko wtedy, gdy odpowiadające im układy dodatnie są praktycznie stabilne, 2) dodatnie układy ułamkowe są praktycznie niestabilne jeżeli odpowiadające im standardowe dodatnie układy ułamkowe są asymptotycznie niestabilne. Sformułowano również proste warunki na stabilność asymptotyczną. Rozważania zostały zobrazowane przykładami numerycznymi.

PRACTICAL STABILITY AND ASIMPTOTIC STABILITY OF CONE FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

A new concept (notion) of the practical stability and asymptotic stability of cone fractional discrete-time linear systems is introduced. Necessary and sufficient conditions for the practical stability and asymptotic stability of the cone fractional systems are established. It is shown that: 1) the cone fractional systems are practically stable if and only if the corresponding positive systems are practically stable, 2) the positive fractional systems are practically unstable if corresponding positive fractional systems are asymptotically unstable. Simple conditions for the asymptotic stability are also established. Considerations are illustrated by numerical example.

1. INTRODUCTION

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems in more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [4, 6].

Mathematical fundamentals of fractional calculus are given in the monographs [17-19, 22]. The fractional positive linear continuous-time and discrete-time systems have been addressed in [8, 10, 20, 21, 26]. Stability of positive 1D and 2D systems has addressed in [13-15, 27, 28] and the robust stability of positive and fractional linear systems has been investigated in [1, 2]. The reachability and controllability to zero of positive fractional linear systems have been considered in [7, 10, 16]. The fractional order controllers have been developed in [22]. A generalization of the Kalman filter for fractional order systems has been proposed in [25]. Fractional polynomials and *n*D systems have been investigated in [5]. The notion of standard and positive 2D fractional linear systems has been introduced in [11, 12].

In this paper a new concept of the practical stability and asymptotic stability of cone fractional discrete-time linear systems will be introduced and necessary and sufficient conditions for the practical stability and asymptotic stability will be established.
The paper is organized as follows. In section 2 the basic definitions and necessary and sufficient conditions for the positivity and asymptotic stability of the linear discrete-time systems are introduced. In section 3 the positive fractional linear discrete-time systems are introduced. The main results of the paper are given in sections 5 and 6 where the concept of practical stability of the cone fractional systems is proposed and necessary and sufficient conditions for the practical stability and asymptotic stability are established. Concluding remarks are given in section 7.
To the best author's knowledge the practical stability and asymptotic stability of the cone fractional systems has not been considered yet.

The following notation will be used in the paper. The set of real $n\times m$ matrices with nonnegative entries will be denoted by $R_+^{n\times m}$ and $R_+^n = R_+^{n\times 1}$. A matrix $A=[a_{ij}]\in R_+^{n\times m}$ (a vector x) will be called strictly positive and denoted by $A>0$ ($x>0$) if $a_{ij}>0$ for $i=1,...,n$ $j=1,...,m$. The set of nonnegative integers will be denoted by Z_+.

2. POSITIVE 1D SYSTEMS

Consider the linear discrete-time system:

$$x_{i+1} = Ax_i + Bu_i, \quad i\in Z_+ \tag{1a}$$

$$y_i = Cx_i + Du_i \tag{1b}$$

where, $x_i\in R^n$, $u_i\in R^m$, $y_i\in R^p$ are the state, input and output vectors and, $A\in R^{nxn}$, $B\in R^{nxm}$, $C\in R^{pxn}$, $D\in R^{pxm}$.

Definition 1. The system (1) is called (internally) positive if $x_i\in R_+^n$, $y_i\in R_+^p$, $i\in Z_+$ for any $x_0\in R_+^n$ and every $u_i\in R_+^m$, $i\in Z_+$.

Theorem 1. [4, 6] The system (1) is positive if and only if

$$A\in R_+^{n\times n},\quad B\in R_+^{n\times m},\quad C\in R_+^{p\times n},\quad D\in R_+^{p\times m}. \tag{2}$$

The positive system (1) is called asymptotically stable if the solution

$$x_i = A^i x_0 \tag{3}$$

of the equation

$$x_{i+1} = Ax_i,\quad A\in R_+^{n\times n},\ i\in Z_+ \tag{4}$$

satisfies the condition

$$\lim_{i\to\infty} x_i = 0 \quad \text{for every } x_0\in R_+^n \tag{5}$$

Theorem 2. [4, 14] For the positive system (4) the following statements are equivalent:
1) The system is asymptotically stable,
2) Eigenvalues $z_1, z_2, \dots, z_n$ of the matrix A have moduli less 1, i.e. $|z_k|<1$ for $k=1,\dots,n$,
3) $\det[I_n z - A]\neq 0$ for $|z|\geq 1$,
4) $\rho(A)<1$ where $\rho(A)$ is the spectral radius of the matrix A defined by $\rho(A)=\max\limits_{1\le k\le n}\{|z_k|\}$
5) All coefficients $\hat{a}_i$, $i=0,1,\dots,n-1$ of the characteristic polynomial

$$p_{\hat{A}}(z) = \det[I_n - \hat{A}] = z^n + \hat{a}_{n-1}z^{n-1} + \ldots + \hat{a}_1 z + \hat{a}_0 \tag{6}$$

of the matrix $\hat{A} = A - I_n$ are positive,

6) All leading principal minors of the matrix

$$\bar{A} = I_n - A = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} \\ \bar{a}_{21} & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n1} & \bar{a}_{n2} & \cdots & \bar{a}_{nn} \end{bmatrix} \tag{7a}$$

are positive, i.e.,

$$|\bar{a}_{11}| > 0, \begin{vmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{vmatrix} > 0, \ldots, \det \bar{A} > 0 \tag{7b}$$

7) There exists a strictly positive vector $\bar{x} > 0$ such that

$$[A - I_n]\bar{x} < 0 \tag{8}$$

Theorem 3. [6] The positive system (4) is unstable if at least one diagonal entry of the matrix A is greater than 1.

3. POSITIVE FRACTIONAL SYSTEMS

In this paper the following definition of the fractional difference

$$\Delta^\alpha x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j}\,, \quad 0 < \alpha < 1 \tag{9}$$

will be used, where $\alpha \in R$ is the order of the fractional difference, and

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \ldots \end{cases} \tag{10}$$

Consider the fractional discrete linear system, described by the state-space equations

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k\,, \quad k \in Z_+ \tag{11a}$$

$$y_k = Cx_k + Du_k \tag{11b}$$

where $x_k \in \Re^n$, $u_k \in \Re^m$, $y_k \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Using the definition (9) we may write the equations (11) in the form

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k\,, \; k \in Z_+ \tag{12a}$$

$$y_k = Cx_k + Du_k \tag{12b}$$

Definition 2. The system (12) is called the (internally) positive fractional system if and only if $x_k \in \Re_+^n$ and $y_k \in \Re_+^p$, $k \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all input sequences $u_k \in \Re_+^m$, $k \in Z_+$.

Theorem 4. The solution of equation (12a) is given by

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} B u_i \tag{13}$$

where Φ_k is determined by the equation

$$\Phi_{k+1} = (A + I_n \alpha)\Phi_k + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1} \tag{14}$$

with $\Phi_0 = I_n$.

The proof is given in [7].

Lemma 1. [8] If

$$0 < \alpha \le 1 \tag{15}$$

then

$$(-1)^{i+1} \binom{\alpha}{i} > 0 \quad \text{for} \quad i = 1, 2, ... \tag{16}$$

Theorem 5. [6] Let $0 < \alpha < 1$. Then the fractional system (12) is positive if and only if

$$A + I_n \alpha \in \Re_+^{n\times n}, \ \ B \in \Re_+^{n\times m}, \ \ C \in \Re_+^{p\times n}, \ \ D \in \Re_+^{p\times m} \tag{17}$$

4. PRACTICAL STABILITY

From (10) and (16) it follows that the coefficients

$$c_j = c_j(\alpha) = (-1)^j \binom{\alpha}{j+1}, \quad j = 1, 2, ... \tag{18}$$

strongly decrease for increasing *j* and they are positive for $0 < \alpha < 1$. In practical problems it is assumed that *j* is bounded by some natural number *h*.
In this case the equation (12a) takes the form

$$x_{k+1} = A_\alpha x_k + \sum_{j=1}^{h} c_j x_{k-j} + B u_k, \quad k \in Z_+ \tag{19}$$

where

$$A_\alpha = A + I_n \alpha \tag{20}$$

Note that the equations (19) and (12b) describe a linear discrete-time system with *h* delays in state.

Definition 3. The positive fractional system (12) is called practically stable if and only if the system (19), (12b) is asymptotically stable.

Defining the new state vector

$$\tilde{x}_k = \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-h} \end{bmatrix} \tag{21}$$

we may write the equations (19) and (12b) in the form

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}u_k, \quad k \in Z_+ \tag{22a}$$

$$y_k = \tilde{C}x_k + \tilde{D}u_k \tag{22b}$$

where

$$\tilde{A} = \begin{bmatrix} A_\alpha & c_1 I_n & c_2 I_n & \dots & c_{h-1} I_n & c_h I_n \\ I_n & 0 & 0 & \dots & 0 & 0 \\ 0 & I_n & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_n & 0 \end{bmatrix} \in \Re_+^{\tilde{n}\times\tilde{n}}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \Re_+^{\tilde{n}\times m}$$

$$\tilde{C} = \begin{bmatrix} C & 0 & \dots & 0 \end{bmatrix} \in \Re_+^{p\times\tilde{n}}, \quad \tilde{D} = D \in \Re_+^{p\times m}, \quad \tilde{n} = (1+h)n \tag{22c}$$

To test the practical stability of the positive fractional system (12) the conditions of Theorem 2 can be used to the system (22).

Theorem 6. The positive fractional system (12) is practically stable if and only if one of the following equivalent conditions is satisfied:

1) Eigenvalues $\tilde{z}_k$, $k = 1,\dots,\tilde{n}$ of the matrix $\tilde{A}$ have moduli less than 1, i.e.

$$|\tilde{z}_k| < 1 \text{ for } k = 1,\dots,\tilde{n} \tag{23}$$

2) $\det[I_{\tilde{n}} z - \tilde{A}] \neq 0$ for $|z| \geq 1$,

3) $\rho(\tilde{A}) < 1$ where $\rho(\tilde{A})$ is the spectral radius of the matrix $\tilde{A}$ defined by $\rho(\tilde{A}) = \max_{1\leq k\leq\tilde{n}}\{|\tilde{z}_k|\}$,

4) All coefficients $\tilde{a}_i$, $i = 0,1,\dots,\tilde{n}-1$ of the characteristic polynomial

$$p_{\tilde{A}}(z) = \det[I_n - \tilde{A}] = z^n + \tilde{a}_{n-1} z^{n-1} + \dots + \tilde{a}_1 z + \tilde{a}_0 \tag{24}$$

of the matrix $[\tilde{A} - I_{\tilde{n}}]$ are positive,

5) All leading principal minors of the matrix

$$[I_{\tilde{n}} - \tilde{A}] = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1\tilde{n}} \\ \tilde{a}_{21} & \tilde{a}_{21} & \dots & \tilde{a}_{2\tilde{n}} \\ \dots & \dots & \dots & \dots \\ \tilde{a}_{\tilde{n}1} & \tilde{a}_{\tilde{n}1} & \dots & \tilde{a}_{\tilde{n}\tilde{n}} \end{bmatrix} \tag{25a}$$

are positive , i.e.

$$|\tilde{a}_{11}| > 0, \quad \begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{vmatrix} > 0, \dots, \det[I_{\tilde{n}} - \tilde{A}] > 0 \tag{25b}$$

6) There exist strictly positive vectors $\bar{x}_i \in \Re_+^n$, $i = 0,1,\dots,h$ satisfying

$$\bar{x}_0 < \bar{x}_1,\ \bar{x}_1 < \bar{x}_2, \dots, \bar{x}_{h-1} < \bar{x}_h \tag{26a}$$

such that

$$A_\alpha \bar{x}_0 + c_1 \bar{x}_1 + \dots + c_h \bar{x}_h < \bar{x}_0 \tag{26b}$$

Proof. The first five conditions 1) - 5) follow immediately from the corresponding conditions of Theorem 2. Using (8) for the matrix $\tilde{A}$ we obtain

$$\begin{bmatrix} A_\alpha & c_1 I_n & c_2 I_n & \dots & c_{h-1} I_n & c_h I_n \\ I_n & 0 & 0 & \dots & 0 & 0 \\ 0 & I_n & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_n & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_0 \\ \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_{h-1} \\ \bar{x}_h \end{bmatrix} < \begin{bmatrix} \bar{x}_0 \\ \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_h \end{bmatrix} \tag{27}$$

From (27) follow the conditions (26). □

Theorem 7. If the positive fractional system (12) is asymptotically stable then the sum of entries of every row of the adjoint matrix $\mathrm{Adj}[I_{\tilde{n}} - \tilde{A}]$ is strictly positive, i.e.

$$\mathrm{Adj}[I_{\tilde{n}} - \tilde{A}]^{-1} \mathbf{1}_{\tilde{n}} > 0 \tag{28}$$

where $\mathbf{1}_{\tilde{n}} = [1 \ \ 1 \ \ \dots \ \ 1]^T \in \Re_+^{\tilde{n}}$, T denotes the transpose.

Proof. It is well-known [9, 14] that if the system (22) is asymptotically stable then

$$\bar{x} = [I_{\tilde{n}} - \tilde{A}]^{-1} \mathbf{1}_{\tilde{n}} > 0 \tag{29}$$

is its strictly positive equilibrium point for $\tilde{B}u = \mathbf{1}_{\tilde{n}}$. Note that

$$\det[I_{\tilde{n}} - \tilde{A}] > 0 \tag{30}$$

since all eigenvalues of the matrix $[I_{\tilde{n}} - \tilde{A}]$ are positive. The conditions (29) and (30) imply (28). □

Example 1. Check the practical stability of the positive fractional system

$$\Delta^\alpha x_{k+1} = 0.1 x_k, \quad k \in Z_+ \tag{31}$$

for $\alpha = 0.5$ and $h = 2$.

Using (18), (20) and (22c) we obtain

$$c_1 = \frac{\alpha(\alpha - 1)}{2} = \frac{1}{8}, \quad c_2 = \frac{1}{16}, \quad a_\alpha = 0.6$$

and

$$\tilde{A} = \begin{bmatrix} a_\alpha & c_1 & c_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & \frac{1}{8} & \frac{1}{16} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

In this case the characteristic polynomial (24) has the form

$$p_{\tilde{A}}(z) = \det[I_{\tilde{n}}(z+1) - \tilde{A}] = \begin{bmatrix} z + 0.4 & -\frac{1}{8} & -\frac{1}{16} \\ -1 & z+1 & 0 \\ 0 & -1 & z+1 \end{bmatrix} = \\ = z^3 + 2.4z^2 + 1.675z + 0.2125 \tag{32}$$

All coefficients of the polynomial (32) are positive and by Theorem 6 the system is practically stable. Therefore, the system is also practically stable and the condition (28) is stable.
Using (28) we obtain

$$\mathrm{Adj}[I_{\tilde{n}} - \tilde{A}]\mathbf{1}_{\tilde{n}} = \left(\mathrm{Adj}\begin{bmatrix} 0.4 & -\frac{1}{8} & -\frac{1}{16} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.0625 \\ 0.6500 \\ 1.6125 \end{bmatrix}$$

and the condition (28) is satisfied.

Theorem 8. The positive fractional system (12) is practically stable only if the positive system

$$x_{k+1} = A_\alpha x_k, \quad k \in Z_+ \tag{33}$$

is asymptotically stable.

Proof. From (26b) we have

$$(A_\alpha - I_n)\overline{x}_0 + c_1\overline{x}_1 + \ldots + c_h\overline{x}_h < 0 \tag{34}$$

Note that the inequality (34) may be satisfied only if there exists a strictly positive vector $\overline{x}_0 \in \Re_+^n$ such that

$$(A_\alpha - I_n)\overline{x}_0 < 0 \tag{35}$$

since $c_1\overline{x}_1 + \ldots + c_h\overline{x}_h > 0$.

By Theorem 2 the condition (35) implies the asymptotic stability of the positive system (33). □

From Theorem 8 we have the following important corollary.

Corollary 1. The positive fractional system (12) is practically unstable for any finite *h* if the positive system (33) is asymptotically unstable.

Theorem 9. The positive fractional system (12) is practically unstable if at least one diagonal entry of the matrix A_α is greater than 1.

Proof. The proof follows immediately from Theorems 8 and 3. □

Example 2. Consider the autonomous positive fractional system described by the equation

$$\Delta^\alpha x_{k+1} = \begin{bmatrix} -0.5 & 1 \\ 2 & 0.5 \end{bmatrix} x_k, \quad k \in Z_+ \tag{36}$$

for $\alpha = 0.8$ and any finite *h*.

In this case $n = 2$ and

$$A_\alpha = A + I_n\alpha = \begin{bmatrix} 0.3 & 1 \\ 2 & 1.3 \end{bmatrix} \tag{37}$$

By Theorem 9 the positive fractional system is practically unstable for any finite *h* since the entry (2,2) of the matrix (37) is greater than 1.

The same result follows from the condition 5 of Theorem 2 since the characteristic polynomial of the matrix $A_\alpha - I_n$

$$p_{\tilde{A}}(z) = \det[I_{\tilde{n}}(z+1) - A_\alpha] = \begin{bmatrix} z+0.7 & -1 \\ -2 & z-0.3 \end{bmatrix} = z^2 + 0.4z - 2.21$$

has one negative coefficient $\hat{a}_0 = -2.21$.

5. ASIMPTOTIC STABILITY

In this section the practical stability of the positive systems for $h \to \infty$ will be addressed.

Definition 4. The positive fractional system (12) is called asymptotically stable if the system is practically stable for $h \to \infty$.

Lemma 2. If $0 < \alpha < 1$ then

$$\sum_{j=1}^{\infty} c_j = 1 - \alpha \tag{38}$$

where the coefficients c_j are defined by (18).

Proof. It is well-known [24] that $\sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} = 0$. Using this equality and (38) we obtain

$$1 - \alpha + \sum_{j=2}^{\infty} (-1)^j \binom{\alpha}{j} = 1 - \alpha - \sum_{j=1}^{\infty} (-1)^j \binom{\alpha}{j+1} = 1 - \alpha - \sum_{j=1}^{\infty} c_j = 0 . \ \square$$

Theorem 10. The positive fractional system (12) is asymptotically stable if and only if positive system

$$x_{i+1} = (A + I_n) x_i \tag{39}$$

is asymptotically stable.

Proof. It is well-known [3] that the positive system (19) for $h \to \infty$ is asymptotically stable if and only if the positive system

$$x_{i+1} = (A_\alpha + \sum_{j=1}^{\infty} c_j I_n) x_i \tag{40}$$

is asymptotically stable. The positive systems (39) and (40) are equivalent since by (38) and (20)

$$A_\alpha + \sum_{j=1}^{\infty} c_j I_n = A + I_n \alpha + I_n (1 - \alpha) = A + I_n . \ \square$$

Applying to the positive system (39) Theorem 6 we obtain the following theorem.

Theorem 11. The positive fractional system (12) is asymptotically stable if and only if one of the equivalent conditions holds:

1) Eigenvalues $z_1, z_2, ..., z_k$ of the matrix $A + I_n$ have moduli less than 1, i.e. $|z_k| < 1$ for $k = 1, ..., n$,
2) All coefficients of the characteristic polynomial of the matrix A are positive,
3) All leading principal minors of the matrix $-A$ are positive.

Theorem 12. The positive fractional system (12) is unstable if at least one diagonal entry of the matrix A is positive.

Proof. If at least one diagonal entry of the matrix A is positive than at least one diagonal entry of the matrix $A + I_n$ is greater than 1 and it is well-known [6, 14] that the system is unstable. □

Example 3. Using Theorem 11 find values of the coefficient *c* for which the positive fractional system (12) with

$$A = \begin{bmatrix} -0.5 & 1 \\ 0.2 & c \end{bmatrix} \text{ and } \alpha = 0.8 \tag{41}$$

is asymptotically stable.
The fractional system is positive if all entries of the matrix

$$A_\alpha = A + I_n\alpha = \begin{bmatrix} 0.3 & 1 \\ 0.2 & c+\alpha \end{bmatrix} \tag{42}$$

are nonnegative, i.e. $c + \alpha \geq 0$ and $c \geq -\alpha = -0.8$.
Applying the condition 2) of Theorem 11 to the matrix (41) we obtain

$$\det[I_n z - A] = \begin{vmatrix} z+0.5 & -1 \\ -0.2 & z-c \end{vmatrix} = z^2 + (0.5 - c)z - (0.5c + 0.2)$$

and $c < -0.4$. Therefore the fractional system (12) with (41) is positive and asymptotically stable for $-0.8 \leq c < -0.4$. The same result we obtain using the condition 3) of Theorem 11.

6. CONE FRACTIONAL SYSTEMS

Definition 5. [8] Let $P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in R^{n\times n}$ be nonsingular and p_k be the *k*-th ($k = 1,\ldots,n$) its row.

The set

$$\mathcal{P} := \left\{ x \in R^n : \bigcap_{k=1}^{n} p_k x \geq 0 \right\} \tag{43}$$

is called a linear cone generated by the matrix *P*.
In a similar way we may define for the inputs u the linear cone

$$\mathcal{Q} := \left\{ u \in R^m : \bigcap_{k=1}^{m} q_k u \geq 0 \right\} \tag{44}$$

generated by the nonsingular matrix $Q = \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} \in R^{m\times m}$ and for the outputs y, the linear cone

$$\mathcal{V} := \left\{ y \in R^p : \bigcap_{k=1}^{p} v_k y \geq 0 \right\} \tag{45}$$

generated by the nonsingular matrix $V = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} \in R^{p\times p}$.

Definition 6. The fractional system (12) is called ($\mathcal{P}, \mathcal{Q}, \mathcal{V}$) cone fractional system if $x_i \in \mathcal{P}$ and $y_i \in \mathcal{V}$, $i \in Z_+$ for every $x_0 \in \mathcal{P}$, $u_i \in \mathcal{Q}$, $i \in Z_+$.

The ($\mathcal{P}, \mathcal{Q}, \mathcal{V}$) cone fractional system (12) will be shortly called the cone fractional system.

Note that if $\mathcal{P} = R_+^n$, $\mathcal{Q} = R_+^m$, $\mathcal{V} = R_+^n$ then the (R_+^n, R_+^m, R_+^p) cone system is equivalent to the classical positive system [4, 6].

Theorem 13. The fractional system (12) is ($\mathcal{P}, \mathcal{Q}, \mathcal{V}$) cone fractional system if and only if

$$\bar{A} = PAP^{-1} \in R_+^{n\times n},\ \bar{B} = PBQ^{-1} \in R_+^{n\times m},\ \bar{C} = VCP^{-1} \in R_+^{p\times n},\ \bar{D} = VDQ^{-1} \in R_+^{p\times m} \tag{46}$$

Proof. Let

$$\bar{x}_i = Px_i,\ \ \bar{u}_i = Qu_i \ \text{and}\ \ \bar{y}_i = Vy_i,\ i \in Z_+ \tag{47}$$

From definition 5 it follows that if $x_i \in \mathcal{P}$ then $\bar{x}_i \in R_+^n$, if $u_i \in \mathcal{Q}$ then $\bar{u}_i \in R_+^m$ and if $y_i \in \mathcal{V}$ then $\bar{y}_i \in R_+^p$. From (12) and (47) we have

$$\begin{aligned}\bar{x}_{k+1} + \sum_{j=1}^{k+1}(-1)^j \binom{\alpha}{j}\bar{x}_{k-j+1} &= Px_{k+1} + \sum_{j=1}^{k+1}(-1)^j \binom{\alpha}{j} Px_{k-j+1} = PAx_k + PBu_k \\ &= PAP^{-1}\bar{x}_k + PBQ^{-1}\bar{u}_k = \bar{A}\bar{x}_k + \bar{B}\bar{u}_k,\ k \in Z_+\end{aligned} \tag{48a}$$

and

$$\bar{y}_k = Vy_k = VCx_k + VDu_k = VCP^{-1}\bar{x}_k + VDQ^{-1}\bar{u}_k = \bar{C}\bar{x}_k + \bar{D}\bar{u}_k,\ k \in Z_+ \tag{48b}$$

It is well-known [6] that the system (48) is the positive one if and only if the conditions (46) are satisfied. □

Theorem 14. The cone fractional system (12) is asymptotically stable if and only if the positive fractional system is asymptotically stable.

Proof. From (46) we have

$$\begin{aligned}\det[Iz - \bar{A}] &= \det[Iz - PAP^{-1}] = \det[P(Iz - A)P^{-1}] \\ &= \det[Iz - A]\det P \det P^{-1} = \det[Iz - A]\end{aligned} \tag{49}$$

since $\det P \det P^{-1} = 1$. □

From Theorem 14 we have the following important corollary.

Corollary 2. The cone fractional system (12) is practically stable (asymptotically stable) if and only if the positive fractional system is practically stable (asymptotically stable).

To test the practical stability and the asymptotic stability of the cone fractional system the Theorem 5 and 6 can be used.

7. CONCLUDIN REMARKS

The new concept (notion) of the practical stability of the cone fractional discrete-time linear systems has been introduced. Necessary and sufficient conditions for the practical stability and the asymptotic stability of the cone fractional systems have been established. It has been shown that: 1) the cone fractional systems are practically stable and the asymptotic stability if and only if the corresponding positive systems are practically stable and the asymptotic stability, 2) the cone fractional system (12) is practically unstable for any finite h if the standard positive system (33) is asymptotically unstable. The considerations have been illustrated by two numerical examples.

The considerations can be easily extended for two-dimensional cone fractional linear systems. An extension of these considerations for continuous-time cone fractional linear systems is an open problem.

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