# Numerical Analysis of the Discrete, Fractional Order PID Controller using FOBD and CFE Approximations

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**Abstract:** This paper presents the numerical analysis of the discrete, approximated Fractional Order PID Controller (FOPID). The fractional parts of the controller are approximated with the use of the most known methods: Fractional Order Backward Difference (FOBD) and Continuous Fraction Expansion (CFE). CFE is simpler and faster than the FOBD method, but its accuracy is not always satisfying. For both approximations optimum sample time was found by minimizing of the cost function Integral Absolute Error (IAE). Additionally, to optimize of CFE its parameter a was applied. Results of numerical tests show that the FOPID using FOBD is more accurate in the sense of IAE cost function for FOPI and FOPID controllers, but CFE is more accurate for FOPD controller. Next, the FOBD requires to use of smaller sample time to obtain of good accuracy than CFE. This allows to conclude that FOPD controller using CFE can be applied in time critical applications at bounded platforms, for example in robotics or numerical control.

**Keywords:** FOPID controller, CFE approximation, FOBD approximation, accuracy, IAE cost function, numerical complexity

## Acronyms

CFE – Continuous Fraction Expansion
FO – Fractional Order
FOPID – Fractional Order PID
FOPI – Fractional Order PI
FOPD – Fractional Order PD
FOBD – Fractional Order Backward Difference
PLC – Programmable Logic Controller
PSE – Power Series Expansion

## 1. Introduction

One of main areas of application fractional order calculus in automation is a FOPID control. Results presented by many Authors, e.g. [2, 4, 14, 15], show that FOPID controller is able to assure better control performance than its integer order PID analogue.

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Each digital implementation of FOPID controller (PLC, microcontroller) requires to apply integer order, finite dimensional, discrete approximant. The most known are: FOBD and CFE approximations (see e.g. [1]). They allow to estimate a non-integer order element with the use of a digital filter. The detailed comparison of both methods was done e.g. in [8]. The use of these methods in the FOPID controller were also considered in the paper [12], the FOPID employing FOBD is analysed with details in the paper [10].

For elementary fractional-order integrator/differentiator an analytical formula of the step response is known (see e.g. [4]). Consequently the analytical step response of a FOPID controller can be given too. It is applied as the reference to estimate of an accuracy of an approximation.

This paper compares the accuracy and numerical complexity of the discrete implementations of the FOPID controller employing the FOBD and CFE approximations. Such a comparison has not been done yet. The approach used in this paper has been proposed in the paper [9]. Recent results can be useful during implementation of FOPID at bounded digital platform.

The paper is organized as follows. Preliminaries draw theoretical background to presenting of main results. It covers basic ideas from fractional calculus as well as the FOPID controller and its discrete form written with the use of both considered approximations.

The numerical analysis covers founding of sample time assuring the best accuracy of approximation and estimating of numerical complexity of both approximations. All versions of controller: FOPID, FOPI and FOPD were examined in the sense accuracy and numerical complexity.

## 2. Preliminaries

### 2.1. Elementary ideas

Elementary ideas from fractional calculus can be found in many books, for example [5, 6, 14, 16]. Here only some definitions necessary to explain of main results are recalled.

Firstly the fractional-order, integro-differential operator is given (see e.g. [5, 7, 16]):

**Definition 1.** (*The elementary fractional order operator*) *The fractional-order integro-differential operator is defined as follows*:

$$ {}_aD_t^\alpha f(t) = \begin{cases} \dfrac{d^\alpha f(t)}{dt^\alpha} & \alpha > 0 \\ f(t) & \alpha = 0 \\ \int_a^t f(\tau)(d\tau)^\alpha & \alpha < 0 \end{cases} \tag{1} $$

where a and t denote time limits for operator calculation, $\alpha \in \mathbb{R}$ denotes the non-integer order of the operation.

Next remember an idea of Gamma Euler function [7]:

**Definition 2.** (*The Gamma function*)

$$ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{2} $$

Furthermore recall an idea of Mittag-Leffler functions. The two parameter Mittag-Leffler function is defined as follows:

**Definition 3.** (*The two parameter Mittag-Leffler function*)

$$ E_{\alpha,\beta}(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(k\alpha + \beta)}. \tag{3} $$

For $\beta = 1$ we obtain the one parameter Mittag-Leffler function:

**Definition 4.** (*The one parameter Mittag-Leffler function*)

$$ E_\alpha(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(k\alpha + 1)}. \tag{4} $$

The fractional-order, integro-differential operator (1) can be described by many definitions. The "classic" have been proposed by Grünwald and Letnikov (GL Definition), Riemann and Liouville (RL Definition) and Caputo (C Definition). In this paper C and GL definition will be employed. They are recalled beneath [4, 13].

**Definition 5.** (*The Caputo definition of the FO operator*)

$$ {}_0^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^\infty \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \tag{5} $$

where $n - 1 < \alpha <$ denotes the non-integer order of operation and $\Gamma(..)$ is the complete Gamma function expressed by (2).

For the Caputo operator the Laplace transform can be defined (see for example [6]):

**Definition 6.** (*The Laplace transform of the Caputo operator*)

$$ \mathcal{L}\left({}_0^CD_t^\alpha f(t)\right) = s^\alpha F(s), \quad \alpha < 0 $$

$$ \mathcal{L}\left({}_0^CD_t^\alpha f(t)\right) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} {}_0D_t^k f(0), \tag{6} $$
$$ \alpha > 0,\ n - 1 < \alpha \le n \in N. $$

**Definition 7.** (*The Grünwald-Letnikov definition of the FO operator*)

$$ {}_0^{GL}D_t^\alpha f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{l=0}^{\left[\frac{t}{h}\right]} (-1)^l \binom{\alpha}{l} f(t - lh). \tag{7} $$

In (7) $\binom{\alpha}{l}$ is the binomial coefficient:

$$ \binom{\alpha}{l} = \begin{cases} 1, & l = 0 \\ \dfrac{\alpha\ \ \alpha-1\ \ldots\ \alpha-l+1}{l!}, & l > 0 \end{cases}. \tag{8} $$

### 2.2. The FOPID controller

The FOPID controller is described by the following transfer function (see e.g. [4], p. 33):

$$ G_c(s) = k_P + k_I s^{-\alpha} + k_D s^\beta, \tag{9} $$

where $\alpha, \beta \in \mathbb{R}$ are fractional orders of the integration and derivative actions and $k_P$, $k_I$ and $k_D$ are the coefficients of the proportional, integral and derivative actions respectively.

The analytical formula of the step response of the controller (9) takes the following form (see [11]):

$$ y_a(t) = k_P + \frac{k_I t^\alpha}{\Gamma(\alpha+1)} + \frac{k_D t^{-\beta}}{\Gamma(1-\beta)}. \tag{10} $$

where $\Gamma(..)$ is the complete Gamma function (2). This analytical formula will be used as the reference to estimate of the accuracy of the approximation.

### 2.3. The FOBD approximation

The GL definition is limit case for $h \to 0$ of the Fractional Order Backward Difference (FOBD), commonly employed in discrete FO calculations (see e.g. [14], p. 68):

**Definition 8.** (The Fractional Order Backward Difference-FOBD)

$$ FOBD = (h^\alpha x)(t) = \frac{1}{h^\alpha} \sum_{l=0}^{L} (-1)^l \binom{\alpha}{l} x(t - lh). \tag{11} $$

Denote coefficients $(-1)^l \binom{\alpha}{l}$ by $d_l(\alpha)$:

$$ d_l(\alpha) = (-1)^l \binom{\alpha}{l}. \tag{12} $$

The coefficients (12) are functions of order $\alpha$. They can be also calculated with the use of the following, equivalent recursive formula (see e.g. [4], p. 12 ), useful in numerical calculations:

$$d_0(\alpha) = 1,$$
$$d_l(\alpha) = \left(1 - \frac{1+\alpha}{l}\right) d_{l-1}(\alpha), \quad l = 1, \ldots, L. \tag{13}$$

It is proven in [3] that:

$$\sum_{l=1}^{\infty} d_l(\alpha) = 1 - \alpha. \tag{14}$$

From (13) and (14) we obtain at once that:

$$\sum_{l=2}^{\infty} d_l(\alpha) = 1. \tag{15}$$

In (11) $L$ denotes a memory length necessary to correct approximation of a non-integer order operator. Unfortunately good accuracy of approximation requires to use a long memory $L$ what can make difficulties during implementation.

The approximator FOBD (11) can be described by the $G(z^{-1})$ transfer function in the form of the FIR filter containing only zeros:

$$G_{FOBD}\left(z^{-1}, \alpha\right) = \frac{1}{h^{\alpha}} \sum_{l=0}^{L} d_l(\alpha) z^{-l}, \tag{16}$$

where $d_l(\alpha)$ are expressed by (12) or equivalently by (13), $h$ is the sample time and $\alpha$ is the fractional order. The transfer function (16) is typically applied to approximate of the fractional operator (1).

## 2.4. The CFE approximation

The CFE approximator allows to express the elementary FO operator $s^{\alpha}$ in the form of an IIR filter containing both poles and zeros. It is faster convergent and easier to implement due to its relatively low order. It is obtained via discretization of elementary fractional order element $s^{\alpha}$. This can be done using so called generating function $s \approx \omega(z^{-1})$. The new operator raised to the power $\alpha$ has the following form [18, 15]:

$$\begin{aligned} \left(\omega(z^{-1})\right)^{\alpha} &= g_h CFE\left\{\left(\frac{1-z^{-1}}{1+az^{-1}}\right)^{\alpha}\right\}_{L,L} = \\ &= g_h \frac{CFE_N(z^{-1}, \alpha)}{CFE_D(z^{-1}, \alpha)} = \\ &= g_h \frac{\sum_{i=0}^{L} w_l z^{-1}}{\sum_{i=0}^{L} v_l z^{-1}} = g_h \frac{P_{\alpha}(z^{-1})}{Q_{\alpha}(z^{-1})}. \end{aligned} \tag{17}$$

In (17) $L$ is the order of approximation, $g_h$ is the coefficient depending on sample time and type of approximation:

$$g_h = \left(\frac{1+a}{h}\right)^{\alpha}. \tag{18}$$

In (18) h is the sample time and a is the coefficient depending on approximation type. For $a = 0$ and $a = 1$ we obtain the Euler and Tustin approximations respectively. For $a \in (0, 1)$ we arrive at the Al-Alaoui-based approximation, which is a linear combination of the Euler and Tustin approximants.

It is important to note that coefficient $a$ can be also employed as an additional parameter allowing to fit the approximation to analytical results, e.g. to obtain accurate approximation of FOPID. This idea will be presented in next sections.

Numerical values of coefficients $w_l$ and $v_l$ and various values of the parameter $a$ can be calculated with use of the MATLAB function *dfod1*. If the Tustin approximation is considered ($a = 1$) then $CFE_D(z^{-1}, \alpha) = CFE_N(z^{-1}, -\alpha)$ and the polynomial $CFE_D(z^{-1}, \alpha)$ can be given in the direct form [18]. Examples of the polynomial $CFE_D(z^{-1}, \alpha)$ for $L = 1, 3, 5$ are given in Table 1. The detailed analysis of various forms of CFE approximators has been presented by [17].

## 2.5. The discrete FOPID using FOBD and CFE approximations

The discrete implementation of the controller (9) using approximator (16) is as beneath:

$$G_{cFOBD}\left(z^{-1}\right) = k_P + k_I G_{FOBD}\left(z^{-1}, -\alpha\right) + k_D G_{FOBD}\left(z^{-1}, -\beta\right). \tag{19}$$

The step response of the approximated controller (19) takes the following form:

$$y_{FOBD}(k) = Z^{-1}\left\{\frac{1}{1-z^{-1}} G_{cFOBD}(z^{-1})\right\}. \tag{20}$$

The formula (20) can be computed numerically with the use of step function from MATLAB. It is the function of a time and memory length $L$. The memory length determines also the accuracy of the implementation. The accuracy and numerical complexity as a function of $L$ were analyzed with details in the paper [10].

**Tab. 1. Coeffcients of polynomials $CFE_{N,D}$ $(z^{-1}, \alpha)$ for Tustin approximation**
Tab. 1. Współczynniki wielomianów $CFE_{N,D}$ $(z^{-1}, \alpha)$ dla aproksymacji Tustina

| Order $L$ | $w_l$ | $v_l$ |
|---|---|---|
| $L = 1$ | $w_1 = -\alpha$<br>$w_0 = 1$ | $v_1 = \alpha$<br>$v_0 = 1$ |
| $L = 3$ | $w_3 = -\alpha/3$<br>$w_2 = \alpha^2/3$<br>$w_1 = -\alpha$<br>$w_0 = 1$ | $v_3 = \alpha/3$<br>$v_2 = \alpha^2/3$<br>$v_1 = \alpha$<br>$v_0 = 1$ |
| $L = 5$ | $w_5 = -\alpha/5$<br>$w_4 = \alpha^2/5$<br>$w_3 = -\left(\frac{\alpha}{5} + \frac{2\alpha^3}{35}\right)$<br>$w_2 = 2\alpha^2/5$<br>$w_1 = -\alpha$<br>$w_0 = 1$ | $v_5 = \alpha/5$<br>$v_4 = \alpha^2/5$<br>$v_3 = -\left(\frac{-\alpha}{5} + \frac{-2\alpha^3}{35}\right)$<br>$v_2 = 2\alpha^2/5$<br>$v_1 = \alpha$<br>$v_0 = 1$ |

Next, the application of (17) to (9) yields:

$$G_{cCFE}(z^{-1}) = k_P + k_I \frac{P_{-\alpha}(z^{-1})}{Q_{-\alpha}(z^{-1})} + k_D \frac{P_\beta(z^{-1})}{Q_\beta(z^{-1})}. \quad (21)$$

The step response of the discrete controller is as follows:

$$y_{CFE}(k) = Z^{-1}\left\{\frac{1}{1-z^{-1}} G_{cCFE}(z^{-1})\right\}. \quad (22)$$

The accuracy of both approximations will be tested using known IAE cost function, calculated at the discrete time grid and for finite time interval.

For the FOBD approximation the cost function is a function of sample time $h$ and memory length $L$ (see [10]). In this paper the minimum memory length equal $L = 100$ will be used and the approximation will be optimized with the use of sample time $h$ only.

$$IAE_{FOBD}(h) = h\sum_{k=1}^{K_f} |e(k)|. \quad (23)$$

For the CFE approximation it is a function of sample time $h$ and parameter $a$, because the length of approximation should be used maximum permitted $M = 5$:

$$IAE_{CFE}(a,h) = h\sum_{k=1}^{K_f} |e(k)|. \quad (24)$$

where $k = 1, ..., K_f$ are the discrete time instants, $h$ is the sample time. Consequently the final time of computing is equal:

$$t_f = hK_f. \quad (25)$$

Error $e(k)$ describes the difference between analytical and approximated step responses (10) vs (20), (22) in the same time moment $k$:

$$e_{FOBD,CFE}(k) = y_a(kh) - y_{FOBD,CFE}(k), \quad k = 1,\ldots,K_f. \quad (26)$$

# 3. Numerical tests

The general methodology of numerical tests is close to approach proposed in the paper [9]. A new idea is the looking for the sample time h assuring the best accuracy of approximation for fixed, short memory length. For CFE approximation additionally its optimum parameter $a$ is estimated too. This is done by minimization of the cost function (23) as a function of h and cost function (24) as a function of $h$ and $a$.

## 3.1. Accuracy

Firstly the approximation using FOBD for $L = 100$ and various versions of FOPID and fractional orders was tested. Calculations were executed at the MATLAB platform using the function *step* to compute of the step response and function *fminbnd* to find optimum value of the sample time $h$. Results are given in the table 2. Rows 1–4 describe the FOPID controller, rows 5–8 illustrate the FOPI controller and rows 9–12 the FOPD controller respectively.

Next the FOPID using CFE was optimized. The values of parameter a and sample time $h$ optimum in the sense of cost function (24) were tested for various orders of FOPID. Calculations were executed at the MATLAB platform using the function

**Tab. 2. Approximation FOBD: optimum values of sample time *h* for various fractional orders of the controller**

Tab. 2. Tab. 2. Aproksymacja FOBD: optymalne wartości okresu próbkowania *h* dla różnych rzędów regulatora

| No. | $\alpha$ | $\beta$ | $h$ | $IAE$ |
|---|---|---|---|---|
| 1 | –0.25 | 0.25 | 0.0758 | 0.0441 |
| 2 | –0.50 | 0.50 | 0.0685 | 0.1543 |
| 3 | –0.75 | 0.75 | 0.0562 | 0.2783 |
| 4 | –0.95 | 0.95 | 0.0365 | 0.2787 |
| 5 | –0.25 | 0.00 | 0.0885 | 0.0776 |
| 6 | –0.50 | 0.00 | 0.0752 | 0.1825 |
| 7 | –0.75 | 0.00 | 0.0621 | 0.2930 |
| 8 | –0.95 | 0.00 | 0.0370 | 0.2753 |
| 9 | 0.00 | 0.25 | 0.0962 | 0.0419 |
| 10 | 0.00 | 0.50 | 0.0971 | 0.0507 |
| 11 | 0.00 | 0.75 | 0.0971 | 0.0279 |
| 12 | 0.00 | 0.95 | 0.0990 | 0.0020 |

**Tab. 3. Approximation CFE: optimum values of parameter *a* and sample time *h* for various fractional orders of controller**

Tab. 3. Tab. 3. Aproksymacja CFE: optymalne wartości parametru *a* oraz okresu próbkowania *h* dla różnych rzędów ułamkowych regulatora

| No. | $\alpha$ | $\beta$ | $a$ | $h$ | $IAE$ |
|---|---|---|---|---|---|
| 1 | –0.25 | 0.25 | 0.0505 | 0.2174 | 0.0888 |
| 2 | –0.50 | 0.50 | 0.0670 | 0.2041 | 0.3375 |
| 3 | –0.75 | 0.75 | 0.0262 | 0.1587 | 0.6404 |
| 4 | –0.95 | 0.95 | 0.0115 | 0.1010 | 0.6577 |
| 5 | –0.25 | 0.00 | 0.0017 | 0.2439 | 0.1757 |
| 6 | –0.50 | 0.00 | 0.0361 | 0.2083 | 0.4271 |
| 7 | –0.75 | 0.00 | 0.1170 | 0.1724 | 0.6862 |
| 8 | –0.95 | 0.00 | 0.1836 | 0.1087 | 0.6289 |
| 9 | 0.00 | 0.25 | –0.0483 | 0.1695 | 0.0390 |
| 10 | 0.00 | 0.50 | –0.2661 | 0.7693 | 0.0254 |
| 11 | 0.00 | 0.75 | –0.0483 | 0.4547 | 0.0888 |
| 12 | 0.00 | 0.95 | –0.0015 | 0.3705 | 0.0014 |

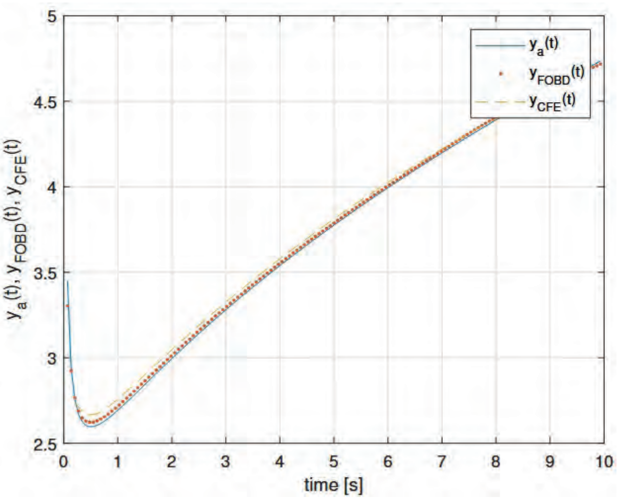


**Fig. 1. The step responses $y_a(t)$ vs $y_{FOBD}(kh)$ and $y_{CFE}(kh)$ for experiment No 2 (controller FOPID)**

Rys. 1. Odpowiedzi skokowe $y_a(t)$ vs $y_{FOBD}(kh)$ i $y_{CFE}(kh)$ dla eksperymentu nr 2 (regulator FOPID)

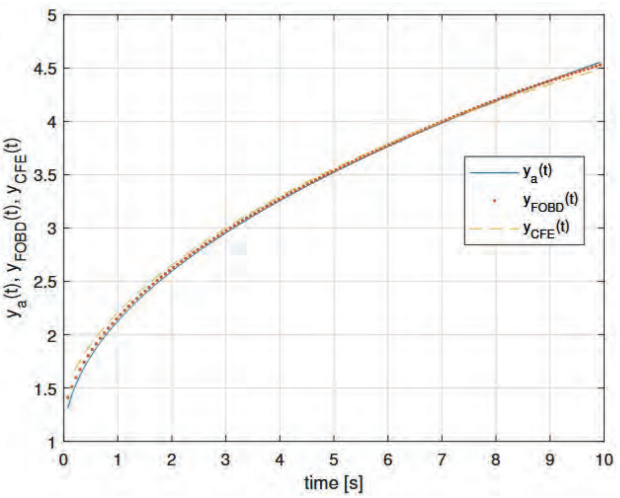


**Fig. 2. The step responses $ya(t)$ vs $y_{FOBD}(kh)$ and $y_{CFE}(kh)$ for experiment No 6 (controller FOPID)**

Rys. 2. Odpowiedzi skokowe $y_a(t)$ vs $y_{FOBD}(kh)$ i $y_{CFE}(kh)$ dla eksperymentu nr 6 (regulator FOPID)

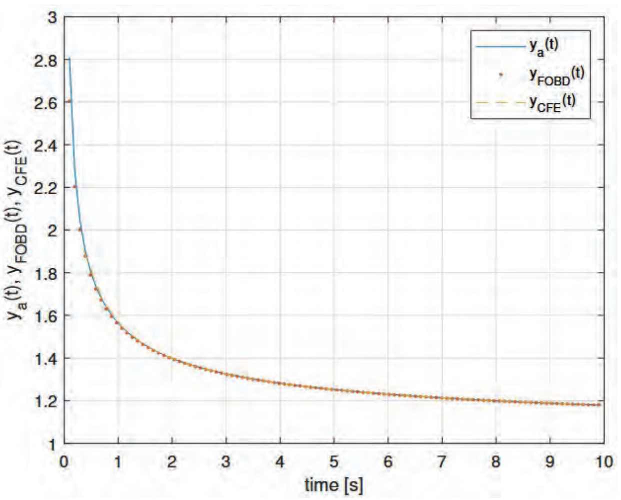


**Fig. 3. The step responses $y_a(t)$ vs $y_{FOBD}(kh)$ and $y_{CFE}(kh)$ for experiment No 10 (controller FOPID)**

Rys. 3. Odpowiedzi skokowe $y_a(t)$ vs $y_{FOBD}(kh)$ i $y_{CFE}(kh)$ dla eksperymentu nr 10 (regulator FOPID)

*step* to compute of the step response and function *fminsearch* to found optimum values of parameters. In all tests the values of parameters $k_P$, $k_I$ and $k_D$ were equal 1.0. Results are collected in the table 3. Rows 1–4 describe the FOPID controller, rows 5–8 illustrate the FOPI controller and rows 9–12 the FOPD controller respectively.

The approximated step responses from experiments No. 2 (FOPID), 6 (FOPI) and 10 (FOPD) are compared to the analytical one in Figures 1–3.

Comparison of step responses analytical vs approximated for experiments No. 2 (FOPID), 6 (FOPI) and 10 (FOPD) is illustrated by Figures 1–3.

The analysis of the tables 2 and 3 allows to conclude that the FOBD gives more accurate approximation for FOPID and FOPI controllers, but for FOPD controller the use of CFE assures better accuracy.

In addition, the good accuracy of the FOPID employing the FOBD requires to apply of shorter sample time, than CFE.

## 3.2. The numerical complexity

Next the numerical complexity was examined. Tests consist in measuring of duration of computation of the step response for each approximation 1000 times. Step responses were calculated using MATLAB function *step*, their duration $T_{FOBD,CFE}$ was measured with the use of MATLAB functions *tic* and *toc*. Tests were done at two PC computers and using various versions of MATLAB. Parameters of the computers used to experiments are given in the Table 4. Experiments were done for parameters sets No. 2, 6 and 10 from Tables 2 and 3. The mean values of $T_{FOBD}$, $T_{CFE}$ and standard deviations $\sigma_{FOBD}$, $\sigma_{CFE}$ obtained from each test are collected in the Table 5. Results presented in the Table 5 show that the calculations executed at the 7 core computer are a little bit faster than at 5 core computer for FOBD approximation only. For CFE this difference is negligible. This is caused by the fact that the order of the CFE model is significantly lower that order of the FOBD model.

Next, the duration of calculations does not depend on the version of controller – the "full" version with integration and derivation is computed in the same time as "reduced" versions FOPI and FOPD. This is caused by the fact that the order of approximation is the same for each version.

**Tab. 4. The parameters of computers used to experiments**

Tab. 4. Parametry komputerów użytych do eksperymentów

| Computer No. 1 | |
|---|---|
| Parameter | |
| Processor | Intel(R) Core(TM) i5-8600K CPU@3.60 GHz |
| RAM | 16 GB |
| OS | Windows 10 Pro |
| MATLAB | R2016b |
| **Computer No. 2** | |
| Parameter | |
| Processor | Intel(R) Core(TM) i7-10700 CPU@2.90 GHz |
| RAM | 16 GB |
| OS | Windows 11 Pro |
| MATLAB | R2020b |

**Tab. 5. Mean values and standard deviations of duration $T_{FOBD}$ and $T_{CFE}$ in seconds from both computers**
Tab. 5. Wartości średnie i odchylenia standardowe czasów obliczeń $T_{FOBD}$ i $T_{CFE}$ dla obu komputerów

| Computer No. 1 | | | | |
|---|---|---|---|---|
| Experiment No | $T_{FOBD}$ | $\sigma_{FOBD}$ | $T_{CFE}$ | $\sigma_{CFE}$ |
| 2 | 2.9359e-04 | 1.4428e-05 | 2.0393e-04 | 1.3842e-05 |
| 6 | 3.0342e-04x | 1.7937e-05 | 2.0696e-04 | 2.4762e-05 |
| 10 | 3.1582e-04 | 3,0884e-05 | 2.0136e-04 | 1.1842e-05 |
| **Computer No. 2** | | | | |
| Experiment No | $T_{FOBD}$ | $\sigma_{FOBD}$ | $T_{CFE}$ | $\sigma_{CFE}$ |
| 2 | 2.4462e-04 | 7.9362e-05 5 | 2.0185e-04 | 1636e-06 |
| 6 | 2.2387e-04 | 1.6100e-05 | 1.9982e-04 | 1.0744e-05 |
| 10 | 1.9537e-04 | 1.2617e-05 | 1.9012e-04 | 1.3107e-05 |

## 4. Final Conclusions

The main final conclusion from this paper is that the selection of approximation FOBD vs CFE should be always carefully considered with respect to accuracy and numerical complexity. For time critical applications using FOPD control at bounded platform the use of CFE seems to be better selection.

The spectrum of further investigations covers e.g. experimental verification of results at industrial digital control platforms: PLC, microcontroller or robot controller.


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## References


1. Al-Alaoui M.A., *Novel digital integrator and differentiator*, "Electronics Letters", Vol. 29, No. 4, 1993, 376–378, DOI: 10.1049/el:19930253.
2. Vinagre B.M., Podlubny I., Hernandez A., Feliu V., *Some approximations of fractional order operators used in control theory and applications*, "Fractional Calculus and Applied Analysis", Vol. 3, No. 3, 2000, 231–248.
3. Busłowicz M., Kaczorek T., *Simple conditions for practical stability of positive fractional discrete-time linear systems*, "International Journal of Applied Mathematics and Computer Science", Vol. 19, No. 2, 2009, 263–269, DOI: 10.2478/v10006-009-0022-6.
4. Caponetto R., Dongola G., Fortuna L., Petras I., *Fractional order systems: Modeling and Control Applications*, 2010, DOI: 10.1142/7709.
5. Das S., *Functional Fractional Calculus for System Identification and Control*, Springer, Berlin 2010.
6. Kaczorek T., *Selected Problems of Fractional Systems Theory*, Springer, Berlin 2011.
7. Kaczorek T., Rogowski K., *Fractional Linear Systems and Electrical Circuits*. Bialystok University of Technology, Bialystok, 2014.
8. Dorcak L., Petráš I., Terpak J., Zbrojovan M., *Comparison of the methods for discrete approximation of the fractional-order operator*. Proceedings of the ICCC'2003 conference, May 26-29, High Tatras, Slovak Republic, 2003, 851–856.
9. Oprzędkiewicz K., *Numerical properties of discrete approximations of an elementary fractional order transfer function*, "Przegląd Elektrotechniczny", R. 99, Nr 7, 2023, 117–123, DOI: 10.15199/48.2023.07.22.
10. Oprzędkiewicz K., *Numerical analysis of the discrete, fractional order PID controller using FOBD approximation*, "Pomiary Automatyka Robotyka", R. 28, Nr 3, 2024, 17–122, DOI: 10.14313/PAR_253/117.
11. Oprzędkiewicz K., *Accuracy estimation of digital fractional order PID controller*, [In:] Theory and applications of non-integer order systems, Lecture Notes in Electrical Engineering, Vol. 407, 2017, 265–275, DOI: 10.1007/978-3-319-45474-0_24.
12. Oprzędkiewicz K., Gawin E., *A non-integer order, state space model for one dimensional heat transfer process*, "Archives of Control Sciences", Vol. 26, No. 2, 2016, 261–275, DOI: 10.1515/acsc-2016-0015.
13. Ostalczyk P., *Equivalent descriptions of a discrete-time fractional-order linear system and its stability domains*, "International Journal of Applied Mathematics and Computer Science", Vol. 22, No. 3, 2012, 533–538, DOI: 10.2478/v10006-012-0040-7.
14. Ostalczyk P., *Discrete Fractional Calculus. Applications in Control and Image Processing*, World Scientific, New Jersey, London, Singapore, 2016.
15. Petráš I., *Fractional-order feedback control of a DC motor*, "Journal of Electrical Engineering", Vol. 60, No. 3, 2009, 117–128.
16. Podlubny I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.
17. Stanisławski R., Latawiec K., Łukaniszyn M., *A Comparative Analysis of Laguerre-Based Approximators to the Grünwald-Letnikov Fractional-Order Difference*, "Mathematical Problems in Engineering", 2015, DOI: 10.1155/2015/512104.
18. Chen Y.Q., Moore K.L., *Discretization schemes for fractional-order differentiators and integrators*, "IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications", Vol. 49, No. 3, 2002, 363–367, DOI: 10.1109/81.989172.

# Analiza numeryczna dyskretnego regulatora PID niecałkowitego rzędu, wykorzystującego aproksymacje FOBD i CFE

Streszczenie: W pracy zaprezentowano analizę numeryczną dyskretnego regulatora PID niecałkowitego rzędu, w którym akcje: całkująca i różniczkująca są aproksymowane z użyciem dwóch typowych aproksymacji dyskretnych: FOBD i CFE. CFE jest szybsza i prostsza, natomiast nie zawsze zapewnia wystarczającą dokładność. Dla obu badanych aproksymacji wyznaczono okres próbkowania zapewniający uch najlepszą dokładność w sensie funkcji kosztu IAE. W przypadku aproksymacji CFE w optymalizacji wykorzystano dodatkowo współczynnik a. Wyniki testów numerycznych wskazują, że zastosowanie aproksymacji FOBD zapewnia lepszą dokładność dla regulatorów FOPID i FOPI, natomiast dla regulatora FOPD lepszą opcją jest zastosowanie CFE. Regulator FOBD dla zapewnienia dobrej dokładności wymaga stosowania krótszego okresu próbkowania, niż CFE. Podsumowując, w krytycznych czasowo aplikacjach pracujących na sprzęcie o ograniczonej mocy obliczeniowej (np. robotyka, sterowanie numeryczne lub urządzenia IoT) można rekomendować zastosowanie regulatora FOPD wykorzystującego aproksymację CFE.

**Słowa kluczowe:** regulator FOPID, aproksymacja CFE, aproksymacja FOBD, dokładność, wskaźnik jakości IAE, złożoność numeryczna

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