# Analysis of chaotic dynamics of the Ikeda system of fractional order

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**Abstract:** The paper considers the Ikeda chaotic system of fractional order. Using numerical simulations effects of fractional order, delay and parameters on chaotic behaviour of the system is investigated. Simulations are performed using Ninteger Fractional Control Toolbox for MATLAB.

**Keywords:** chaos, fractional system, Ikeda system, time-delay.

## 1. Introduction

Dynamical systems described by fractional order differential or difference equations have been investigated in several areas such as viscoelasticity, electrochemistry, diffusion processes, control theory, electrical engineering, etc. The problems of analysis and synthesis of dynamic systems described by fractional order differential (or difference) equations have recently considerable attention, see [1, 3, 7, 10–14], for example.

Many non-linear dynamical systems have behaviour known as chaos. Chaos is a very interesting non-linear phenomenon. Recently it has been intensively studied in many papers and books, see [4, 5, 8, 9, 15], for example, and references therein.

More recently, many investigations are devoted to chaotic dynamics of fractional order dynamical systems, for example [2, 6], Chapters 5 and 6 in [12].

In this paper we consider the Ikeda chaotic system described by the fractional order non-linear differential equation and using numerical simulations we examine effects of fractional order, delay and parameters on chaotic behaviour of the system. Simulations were performed using Ninteger Fractional Control Toolbox for MATLAB [16].

The Ikeda model (standard not fractional) was introduced to describe the dynamics of an optical bistable resonator [4, 5].

## 2. Preliminaries and the main results

Consider the Ikeda time-delay system described by the equation

$$\dot{x}(t) = -ax(t) + b\sin x(t - h(t)), \qquad (1)$$

where $a$, $b$ are constant coefficients and $h$ is the delay.

The Ikeda model was introduced to describe the dynamics of an optical bistable resonator. In this model $x(t)$ is the phase lag of the electric field across the resonator, $a$ is the relaxation coefficient for the dynamical variable, $b$ is the laser intensity injected into the system and $h(t)$ is the round-trip time of the light in the resonator or feedback delay time in the coupled systems [5, 9].

If

$$a = 1, \quad b = 4, \quad h(t) = h = 1.5, \qquad (2)$$

the system (1) is chaotic. Chaotic trajectories of the system for $t \in [0, 200]$ with initial conditions $x_0(\tau) = x_0 = -0.1$ (blue line, 1) and $x_0 = 0.1$ (red line, 2), $\tau \in [-h, 0]$ are shown in fig. 1.

From simulations it follow that if

$$a = 1, \quad b = 3, \quad h = 1.5$$

or

$$a = 1, \quad b = 4, \quad h = 1,$$

the limit cycle behaviour is observed. Trajectory for $a = 1$, $b = 3$, $h = 1.5$ with initial condition $x_0 = 0.1$ is shown in fig. 2. This trajectory tends to a limit cycle.

For $a = 1$, $b = 4$ and $h = 2$ the system (1) has chaotic trajectory. Fig. 3 shows this trajectory with initial condition $x_0 = 0.1$.

In [6] it was shown that if

$$a = 3, \quad b = 24, \quad h(t) = h = 0.2, \qquad (3)$$

the system (1) is chaotic as it is shown in Fig. 4 for $t \in [0, 80]$ with $x_0 = 0.1$.

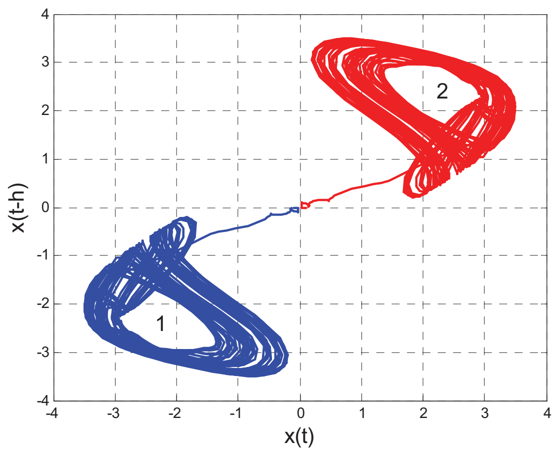


**Fig. 1.** Chaotic trajectories of the system (1), (2) with $x_0 = -0.1$ (blue line, 1) and $x_0 = 0.1$ (red line, 2)

**Rys. 1.** Trajektorie chaotyczne układu (1), (2) przy $x_0 = -0{,}1$ (linia niebieska, 1) oraz $x_0 = 0{,}1$ (linia czerwona, 2)

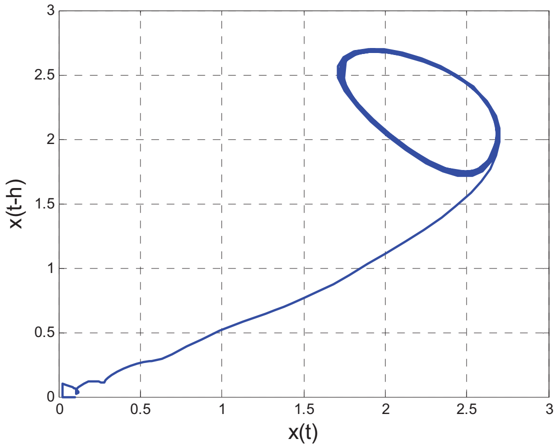


**Fig. 2.** Trajectory of the system (1) for $a=1,\ b=3,\ h=1.5$
**Rys. 2.** Trajektoria układu (1) dla $a=1,\ b=3,\ h=1{,}5$

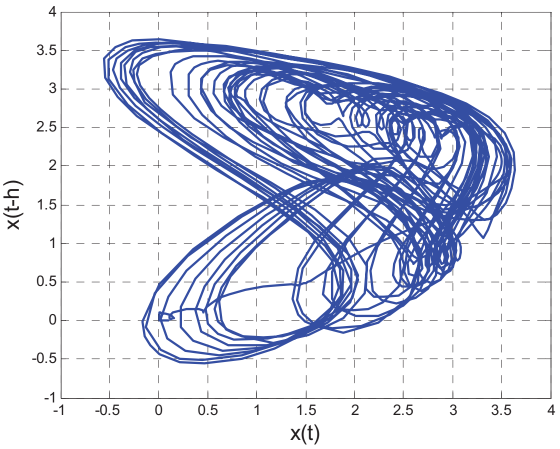


**Fig. 3.** Chaotic trajectory of (1) for $a=1,\ b=4,\ h=2$
**Rys. 3.** Trajektoria chaotyczna układu (1) dla $a=1,\ b=4,\ h=2$

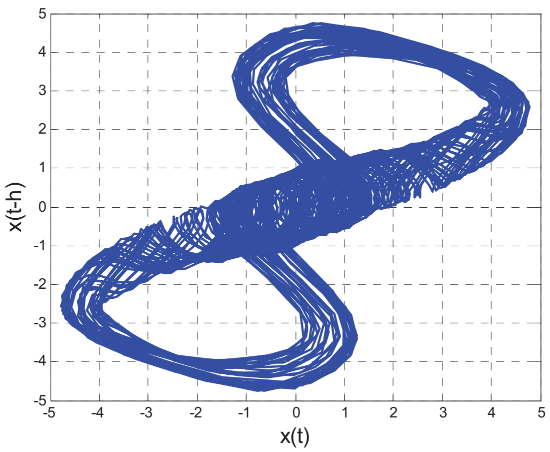


**Fig. 4.** Chaotic trajectory of the system (1), (3)
**Rys. 4.** Trajektoria chaotyczna układu (1), (3)

In this paper we consider the fractional order Ikeda time-delay system described by the equation

$$ {}_0D_t^{\alpha} = -ax(t) + b\sin x(t-h(t)), \qquad (4)$$

where $\alpha$ is the fractional order of derivative satisfying inequality $0<\alpha<2$,

$$ {}_0D_t^{\alpha}x(t) = \frac{1}{\Gamma(p-\alpha)}\int_0^t \frac{x^{(p)}(\tau)d\tau}{(t-\tau)^{\alpha+1-p}}, \quad p-1\le\alpha\le p, \qquad (5)$$

is the Caputo definition for fractional $\alpha$-order derivative, where $x^{(p)}(t)=d^px(t)/dt^p$, $p$ is a positive integer and

$$\Gamma(\alpha)=\int_0^{\infty} e^{-t}t^{\alpha-1}dt \qquad (6)$$

is the Euler gamma function.

From (5) for $p=1$ and $p=2$ we have, respectively,

$$ {}_0D_t^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{x^{(1)}(\tau)}{(t-\tau)^{\alpha}}d\tau, \quad 0<\alpha<1, \qquad (7)$$

$$ {}_0D_t^{\alpha}x(t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{x^{(2)}(\tau)d\tau}{(t-\tau)^{\alpha-1}}, \quad 1<\alpha<2. \qquad (8)$$

The Laplace transform of the Caputo fractional derivative has the form

$$L\{{}_0D_t^{\alpha}x(t)\} = s^{\alpha}F(s) - \sum_{k=1}^{p} s^{\alpha-k}x^{(k-1)}(0^+). \qquad (9)$$

For zero initial conditions, the Laplace transform (9) reduces to

$$L\{{}_0D_t^{\alpha}x(t)\} = s^{\alpha}F(s). \qquad (10)$$

The chaotic dynamics of the system (4), (3) was investigated in [6] for fractional order $0<\alpha<1$. Simulations were performed for $\alpha$ varying from 0.9 to 0.1 with the step $\Delta\alpha=0.1$ and chaotic attractors were found for all these values of the fractional order $\alpha$.

In this paper we consider the fractional Ikeda system (4) with $\alpha\in(0,2)$. In simulations we wary fractional order $\alpha$, parameter $b$ and time delay $h$. Parameter $a=1$ is fixed. For simulation we apply the Ninteger Fractional Control Toolbox for MATLAB of Valerio [16]. In this toolbox exists a Simulink block *nid* for fractional derivative and integral. Order and method for rational approximation of fractional derivative/integral can be selected. In simulations we select the Oustaloup's approximation technique (CRONE) of order $n$=7.

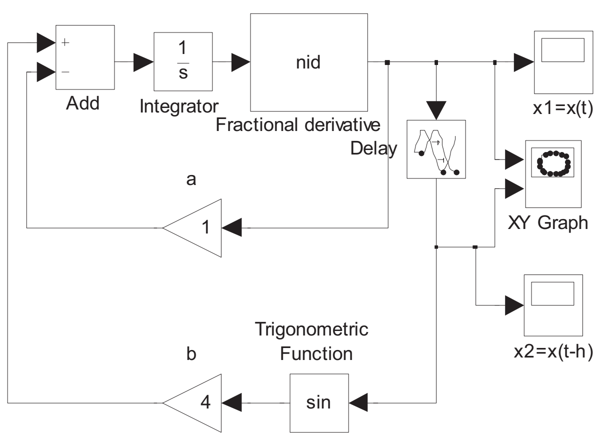


**Fig. 5.** Matlab/Simulink model of the fractional system (4)
**Rys. 5.** Model układu (4) w środowisku MATLAB/Simulink

The block *nid* has the transfer function $ks^v$, where $v$ is a real number.

The model of the fractional system (4) created in the MATLAB/Simulink environment is shown in fig. 5. The fractional integrator $1/s^\alpha$ is modelled by series connection of the classical integrator and the block *nid*. Transfer function of this connection is $k/s^{v-1}$. It is easy to see that $v\in(0,1)$ for $\alpha\in(0,1)$ and $v\in(-1,0)$ for $\alpha\in(1,2)$.

First, we study the effect of fractional order $\alpha\in(0,2)$ on the chaotic behaviour of the system (4) for fixed values $a$, $b$ and $h$, given in (2). Performing simulations we vary fractional order $\alpha$ from 0.1 to 1.9 with the step $\Delta\alpha=0.1$. From simulations it follows that the system (4) with parameters (2) has chaotic behaviour for $\alpha=0.95$ (fig. 7), $\alpha=1$ (fig. 1), $\alpha=1.1 \dots \alpha=1.6$, $\alpha=1.8$ and $\alpha=1.9$ (fig. 9). For $\alpha=0.9$ and $\alpha=1.7$ the limit cycles are observed (figs. 6 and 8).

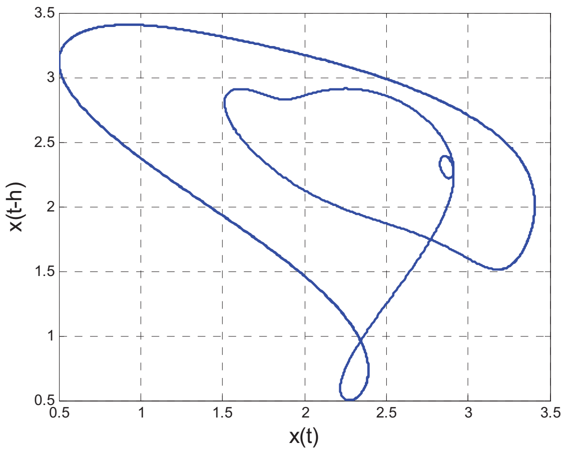


**Fig. 6.** Limit cycle of (4), (2) for $\alpha=0.9$
**Rys. 6.** Cykl graniczny układu (4), (2) dla $\alpha=0{,}9$

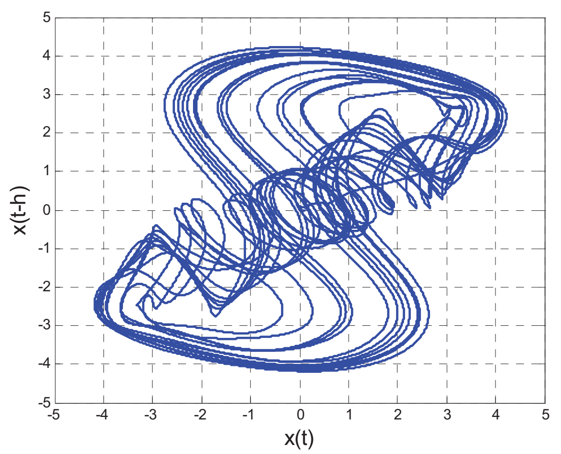


**Fig. 7.** Chaotic trajectory of (4), (2) for $\alpha=0.95$
**Rys. 7.** Trajektoria chaotyczna układu (4), (2) dla $\alpha=0{,}95$

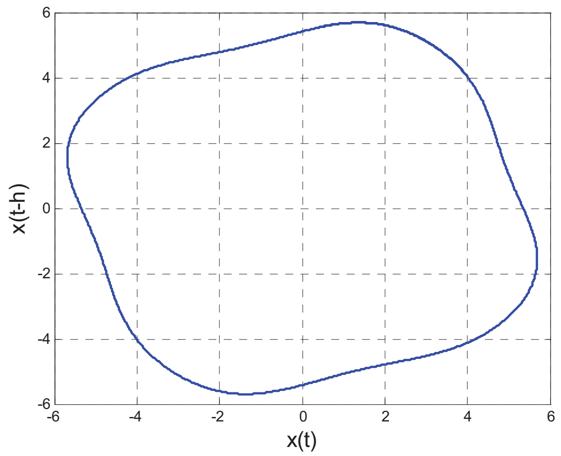


**Fig. 8.** Limit cycle of (4), (2) for $\alpha=1.7$
**Rys. 8.** Cykl graniczny układu (4), (2) dla $\alpha=1{,}7$

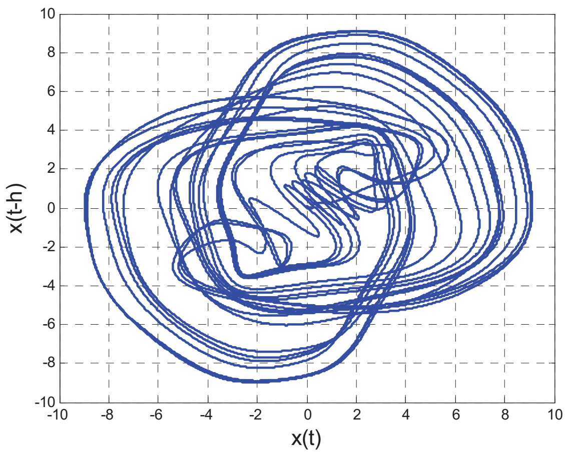


**Fig. 9.** Chaotic trajectory of (4), (2) for $\alpha=1.9$
**Rys. 9.** Trajektoria chaotyczna układu (4), (2) dla $\alpha=1{,}9$

Next, we consider the following two cases:
Case 1: the system (4) with parameters

$$a=1,\quad b=5,\quad h(t)=h=1.5, \tag{11}$$

and the fractional order $\alpha$ varying from 0.1 to 1.9 with the step $\Delta\alpha=0.1$. From simulations it follows that the system has chaotic behaviour for all considered values of $\alpha$. Selected trajectories are shown in figs. 10–12.
Case 2: the system (4) with parameters

$$a=1,\quad b=4,\quad h(t)=h=2. \tag{12}$$

From simulations it follows that for $\alpha=0.6 \dots \alpha=1.9$ the system has chaotic behaviour or limit cycle. The limit cycle is observed for $\alpha=0.6$, $\alpha=1.6$ … $\alpha=1.8$; chaotic behaviour is observed for $\alpha=0.7$ … $\alpha=1.5$ and $\alpha=1.9$. Selected trajectories are shown in figs. 13–20.

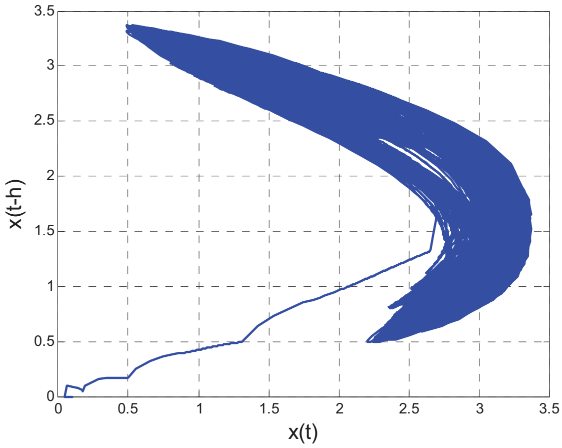


**Fig. 10.** Chaotic trajectory of (4), (11) for $\alpha = 0.1$
**Rys. 10.** Trajektoria chaotyczna układu (4), (11) dla $\alpha = 0{,}1$

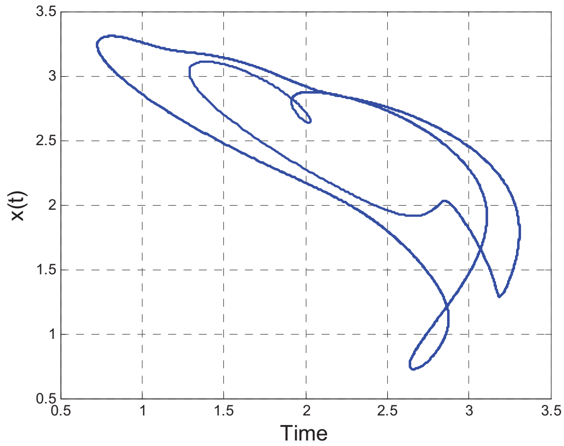


**Fig. 13.** Limit cycle of (4), (12) for $\alpha = 0.6$
**Rys. 13.** Cykl graniczny układu (4), (12) dla $\alpha = 0{,}6$

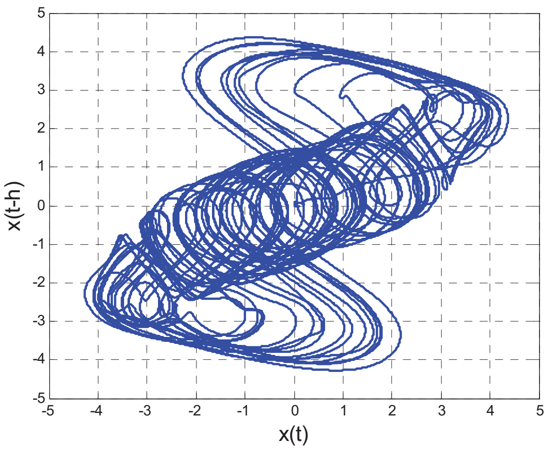


**Fig. 11.** Chaotic trajectory of (4), (11) for $\alpha = 0.9$
**Rys.11.** Trajektoria chaotyczna układu (4), (11) dla $\alpha = 0{,}9$

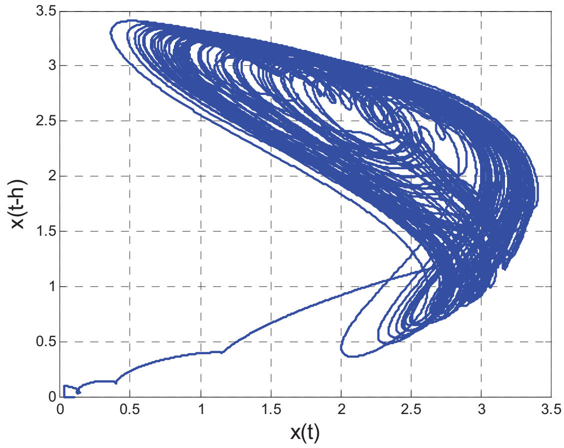


**Fig. 14.** Chaotic trajectory of (4), (12) for $\alpha = 0.7$
**Rys.14.** Trajektoria chaotyczna układu (4), (12) dla $\alpha = 0{,}7$

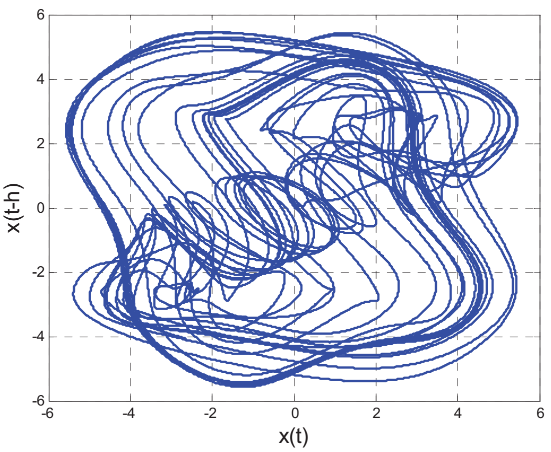


**Fig. 12.** Chaotic trajectory of (4), (11) for $\alpha = 1.5$
**Rys.12.** Trajektoria chaotyczna układu (4), (11) dla $\alpha = 1{,}5$

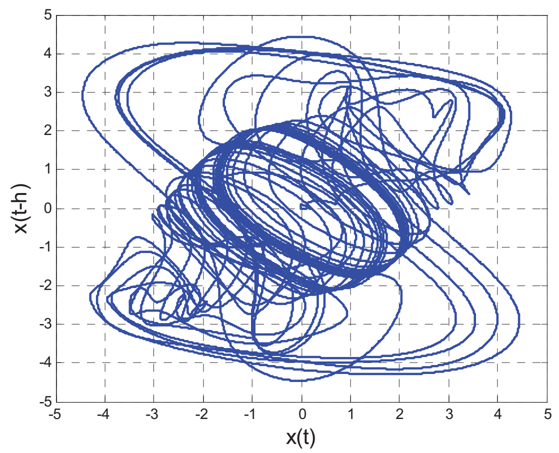


**Fig. 15.** Chaotic trajectory of (4), (12) for $\alpha = 1.5$
**Rys.15.** Trajektoria chaotyczna układu (4), (12) dla $\alpha = 1{,}5$

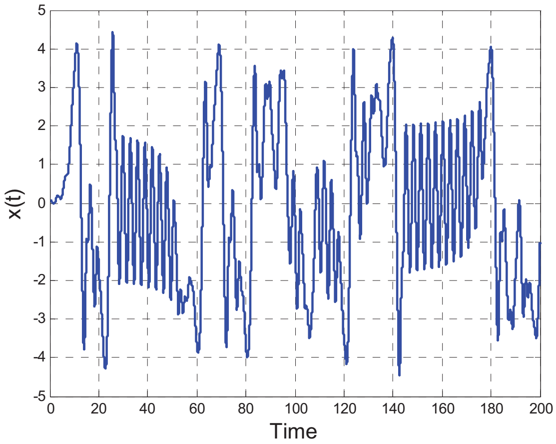


**Fig. 16.** Plot of $x(t)$ for the system (4), (12) for $\alpha = 1.5$
**Rys.16.** Wykres $x(t)$ dla układu (4), (12) dla $\alpha = 1{,}5$

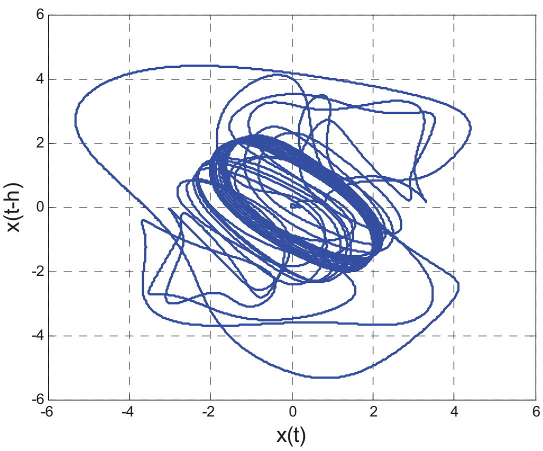


**Fig. 17.** Trajectory of (4), (12) for $\alpha = 1.6$
**Rys.17.** Trajektoria układu (4), (12) dla $\alpha = 1{,}6$

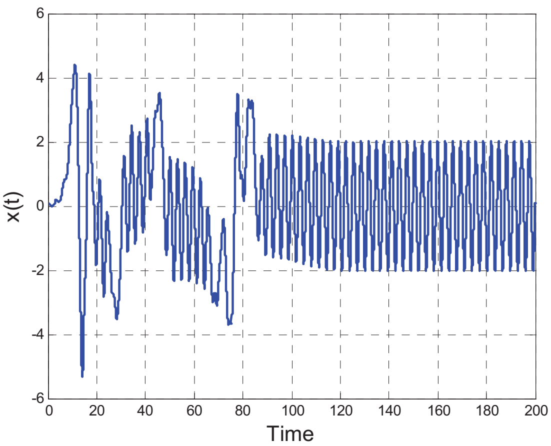


**Fig. 18.** Plot of $x(t)$ for the system (4), (12) for $\alpha = 1.6$
**Rys.18.** Wykres $x(t)$ dla układu (4), (12) dla $\alpha = 1{,}6$

## 3. Concluding remarks

Using numerical simulations, chaotic dynamics of the fractional order Ikeda system (4) has been studied. Simulations have been performed using Ninteger Fractional Control Toolbox for MatLab.

First, it has been shown that the integer order Ikeda system (1) for $a = 1$, $b = 3$, $h = 1.5$ and for $a = 1$, $b = 4$, $h = 1$ has the limit cycles.

Next, for the fractional system (4) it has been concluded that the system has a chaotic behaviour for following values of parameters:

– $a = 1$, $b = 4$, $h = 1.5$ and $\alpha = 0.95$, $\alpha = 1$, $\alpha = 1.1$ ... $\alpha = 1.6$, $\alpha = 1.8$ and $\alpha = 1.9$
– $a = 1$, $b = 5$, $h = 1.5$ and $\alpha = 1.1$ ... $\alpha = 1.9$
– $a = 1$, $b = 4$, $h = 2$ and $\alpha = 0.7$ ... $\alpha = 1.5$ and $\alpha = 1.9$.
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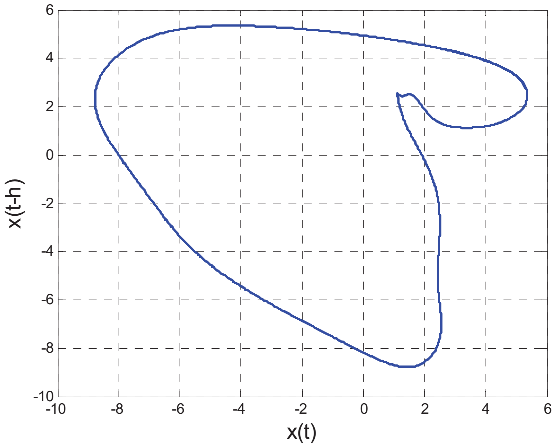


**Fig. 19.** Limit cycle of (4), (12) for $\alpha = 1.8$
**Rys. 19.** Cykl graniczny układu (4), (12) dla $\alpha = 1{,}8$

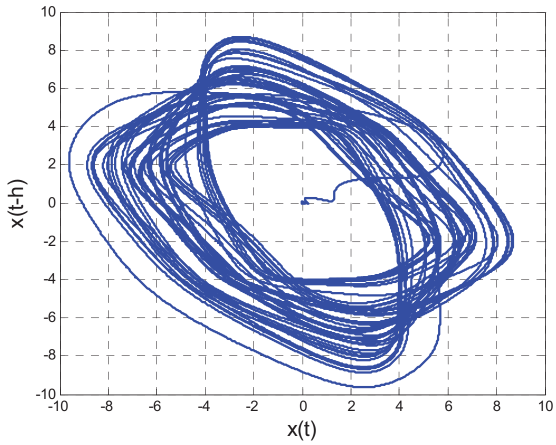


**Fig. 20.** Chaotic trajectory of (4), (12) for $\alpha = 1.9$
**Rys. 20.** Trajektoria chaotyczna układu (4), (12) dla $\alpha = 1{,}9$

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## Analiza chaotycznej dynamiki układu Ikedy niecałkowitego rzędu


**Streszczenie:** Rozpatrzono chaotyczny układ Ikedy niecałkowitego rzędu. Stosując badania symulacyjne zbadano wpływ wartości niecałkowitego rzędu, opóźnienia oraz parametrów układu na możliwość występowania drgań chaotycznych. Badania przeprowadzono w środowisku systemu Matlab/Simulink wykorzystując Ninteger Fractional Control Toolbox for MatLab.



**Słowa kluczowe:** chaos, układ niecałkowitego rzędu, układ Ikedy, opóźnienie.



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