Reachability and controllability of positive fractional-order discrete-time systems

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Abstract: In the paper the positive linear discrete-time non-commensurate fractional-order systems described by the state equations are considered. Definition and necessary and sufficient conditions for the positivity, reachability and controllability to zero are given and proven. The considerations are illustrated by a numerical example.

Keywords: non-commensurate fractional-order, positive, discrete-time systems, reachability, controllability

The fundamental question for dynamic system modeled by state space representation is to determine whether it is possible to transfer state from a given initial state to any other state. The reachability and controllability problems for linear fractional-order state-space system have been studies for some time already.

1. Introduction

The concept of non-integer derivate and integral is increasingly used to model the behavior of real systems in various fields of science and engineering. The mathematical fundamentals of fractional (non-integer) calculus are given in the monographs [11, 12, 18]. This idea has been used by engineers for modelling different process and designing fractional order controllers for time-delay systems [2, 9].

The state-space representation of fractional-order discrete-time system was introduced in [3, 4] and more clear and suitable definitions of reachability, controllability and observability are given. It emerged that for fractional-order system, two different interesting types can be considered: the commensurate order and the non-commensurate order systems. The system is a commensurate order if the differentiation order is taken the same for all the state variable.

In the monograph [6] new classes of commensurate fractional order positive systems: continuous and discrete-time were introduced and necessary and sufficient conditions for reachability and controllability were given. In positive systems inputs, state variables and outputs take only non-negative values for non-negative initial conditions and non-negative controls. Examples of positive systems are given in monograph [7] and quoted there literature.

Positive linear systems are defined on cones and not on linear spaces. Therefore, theory of positive systems is more complicated and less advanced. Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in [1, 7, 14].

In this paper using recent results, given in [3, 4, 5, 6, 8, 16, 17], a problem of reachability and controllability of non-commensurate fractional-order positive discrete-time systems will be considered. The paper is organized as follows. In section 2 using the fractional backward difference the definition of the positive non-commensurate fractional-order discrete time systems is introduced and basic system properties are given as well. For such a system the necessary and sufficient conditions for the reachability and controllability are established in sections 3 and 4, respectively. A numerical example is given in section 5.

2. Linear discrete-time fractional-order systems

Let $\Re^{n\times m}$ be the set of $n\times m$ matrices with entries from the field of real numbers and $\Re^n=\Re^{n\times l}$. The set of $n\times m$ real matrices with nonnegative entries will be denoted by $\Re_+^{n\times m}$, and $\Re_+^n=\Re_+^{n\times l}$. The set of nonnegative integers will be denoted by Z_+, and $n\times n$ identity matrix by I_n.

In this paper the following definition of a generalization fractional order backward difference will be used [6, 11, 12]

$$\Delta^{\alpha_j}x_j(i)=\frac{1}{h^{\alpha_j}}\sum_{k=0}^{i}(-1)^k\binom{\alpha_j}{k}x_j(i-k), \tag{1}$$

where $\alpha_j\in R$ is an order of the fractional difference, h is the sampling interval and $i\in Z_+$ is a number of the sample for which the difference is calculated and the Newton's binomial coefficients can be obtained from

$$\binom{\alpha_j}{k}=\begin{cases}1 & k=0\\ \dfrac{\alpha_j(\alpha_j-1)\cdots(\alpha_j-k+1)}{k!} & k=1,2,\dots\end{cases} \tag{2}$$

According to this definition, it is possible to obtain a discrete equivalent of the derivative (when α_j is positive), a discrete equivalent of the integration (when α_j is negative) and, when $\alpha_j=0$, the original function.

Consider the linear non-commensurate fractional-order discrete-time linear system, described by the state-space equations

$$\Delta^{\alpha}x(i+1)=Ax(i)+Bu(i),\quad i\in Z_+, \tag{3a}$$

$$y(i)=Cx(i)+Du(i), \tag{3b}$$

where

$$\Delta^{\alpha} x(i+1)=\begin{bmatrix}\Delta^{\alpha_1} x_1(i+1)\\ \vdots \\ \Delta^{\alpha_q} x_q(i+1)\end{bmatrix}\in \Re^n, \tag{4a}$$

in which

$$0<\alpha_j<1 \quad \text{for} \quad j=1,...,q\,, \quad q\le n, \tag{4b}$$

denote any fractional orders, and

$$x(i)=\begin{bmatrix}x_1(i)\\ \vdots \\ x_q(i)\end{bmatrix}\in \Re^n, \tag{4c}$$

where $x_j(i)\in \Re^{n_j}$ ($j=1,...,q$) are components of the state vector $x(i)\in \Re^n$, $n=n_1+\cdots n_q$ and

$$A=\begin{bmatrix}A_{11} & \dots & A_{1q}\\ \vdots & \dots & \vdots \\ A_{q1} & \dots & A_{qq}\end{bmatrix}\in \Re^{nxn}, \tag{4d}$$

$A_{kj}\in \Re^{n_k\times n_j}$, $B\in \Re^{n\times m}$, $C\in \Re^{p\times n}$, $D\in \Re^{p\times m}$, $u(i)\in \Re^m$, $y(i)\in \Re^p$,
with the initial condition

$$x_0=x(0)=[x_1(0) \quad \cdots \quad x_q(0)]^T\in \Re^n. \tag{5}$$

Note that for some $\alpha_j=1$, we obtain first-order backward difference and the classical integer-order state-space equation

$$x_j(i+1)=[A_{j1} \quad \cdots \quad A_{jq}]x(i)+x_j(i). \tag{6}$$

This case will be classified as a non-commensurate real-order discrete-time system.

In the case of commensurate fractional-order, the difference order is taken the same for all the state variables $x_j(i)\in \Re^{n_j}$, $j=1,...,q$, i.e.

$$\alpha_1=\alpha_2=\cdots=\alpha_q=\alpha. \tag{7}$$

Then the state-space equation (3a) reduces to the form [3, 6]

$$\Delta^{\alpha} x(i+1)=Ax(i)+Bu(i). \tag{7a}$$

Therefore, a theory of commensurate fractional-order systems is less complicated and more advanced. Some properties of such systems are presented in [3, 6, 11, 12, 17, 18].

Let

$$c_k(\alpha_j)=(-1)^{k+1}\binom{\alpha_j}{k}, \quad k=1,2,\dots \tag{8}$$

where the binomial is given by (2).

Using the definition (1) for $h=1$ we may rewrite the equation (3a) in the form

$$x(i+1)=A_{\overline{\alpha}}x(i)+\sum_{k=2}^{i+1}A_k x(i-k+1), \quad i\in Z_+, \tag{9}$$

where

$$A_{\alpha}=A+\alpha, \tag{9a}$$

$$\overline{\alpha}=diag[\alpha_1 I_{n_1} \quad \cdots \quad \alpha_q I_{n_q}]\in \Re^{nxn} \tag{9b}$$

$$A_k=diag[c_k(\alpha_1)I_{n_1} \quad \cdots \quad c_k(\alpha_q)I_{n_q}]\in \Re^{nxn}, \quad k=2,3,\dots \tag{9c}$$

In the case of non-commensurate real-order, in formulas (9b) and (9c) we substitute for $\alpha_j=1$, respectively

$$\alpha_j I_{n_j}=I_{n_j} \tag{10a}$$

$$c_k(\alpha_j)=0, \quad k=2,3,\dots \tag{10b}$$

In the case of commensurate fractional-order, the system is described by equation (9), where the matrices (9a) and (9c) take the following expressions:

$$A_{\overline{\alpha}}=A+\alpha I_n, \tag{11a}$$

$$A_k=c_k(\alpha)I_n \quad k=2,3,\dots \tag{11b}$$

where coefficients c_k are given by (8) for $0<\alpha<1$.

Note that the fractional discrete-time linear system (9) is the classical discrete-time system with delays increasing with the number of samples $i\in Z_+$ [4, 6]. From (8) it follows that coefficients $c_k(\alpha_j)$, $k=1,2,\dots$ strongly decrease to zero for any fractional orders $0<\alpha_j<1$, $j=1,\dots,q$, when j increases to infinity.

Theorem 1. [6] The solution of equation (3a) with initial conditions (5) is given by

$$x(i)=\Phi_i x_0+\sum_{j=0}^{i-1}\Phi_{i-1-j}Bu(j), \tag{12}$$

where the fundamental (transition) matrix Φ_i is determined by the equation

$$\Phi_{i+1}=(A+\overline{\alpha})\Phi_i+\sum_{k=2}^{i+1}A_k\Phi_{i-k+1} \tag{13}$$

with the initial conditions

$$\Phi_0=I, \quad \Phi_i=0 \quad dla \quad i<0, \tag{14}$$

where matrices $\overline{\alpha}$ and A_k are given by (9b) and (9c).

The proof using the Z transform is similar as is given in [6, 8] in the case of commensurate fractional-order discrete time system.

Note that the solution (12) of fractional state equation can be derived using the recursive formula (9) for $x(i)$, $i=0,1,2,\dots$ and the initial condition (5) without applying the inverse Z transform [4].

Definition 1. [6, 17] The any fractional-order system (3) is called the (internally) positive fractional system if and only if $x(i)\in \Re_+^n$ and $y(i)\in \Re_+^p$, $i\in Z_+$ for any initial conditions $x_0\in \Re_+^n$ and all input sequences $u_i\in \Re_+^m$, $i\in Z_+$.

The following two lemmas will be used in the proof of the positivity of the fractional system (3).

Lemma 1. [6] If the order of the fractional difference α_j satisfies

$$0<\alpha_j<1 \tag{15}$$

then coefficients (8) are positive, i.e. $c_k(\alpha_j) > 0 \quad k = 1, 2, \ldots$

The proof of the lemma is given in [6].

Lemma 2. If the order of the difference α_j satisfies the condition $0 < \alpha_j \le 1$ and

$$A_\alpha = A + \alpha \in \Re_+^{n\times n}, \tag{16}$$

then fundamental matrices (13) have only nonnegative entries, i.e.

$$\Phi_i \in \Re_+^{n\times n}, \quad i \in Z_+. \tag{17}$$

Proof. Using (13) for $i = 1, 2, \ldots$ we obtain fundamental matrices Φ_i of the forms:

$$\Phi_1 = A_{\bar{\alpha}}, \tag{18a}$$

$$\Phi_2 = A_\alpha \Phi_1 + A_2\Phi_0 = A_\alpha^2 + A_2 \tag{18b}$$

$$\Phi_3 = A_\alpha\Phi_2 + A_2\Phi_1 + A_3\Phi_0 = A_\alpha^3 + A_\alpha A_2 + A_2 A_\alpha + A_3, \tag{18c}$$

$$\vdots$$

$$\Phi_q = A_\alpha\Phi_{q-1} + A_2\Phi_{q-2} + \ldots + A_q\Phi_0 = A_\alpha^q + A_2 A_\alpha^{q-2} + \ldots + A_q. \tag{18d}$$

where matrices A_α and A_k are given by (9a) and (9c) in the case of a non-commensurate fractional-order and by (11a)-(11b) in the case of a commensurate fractional-order.

From Lemma 1 and the above it follows that the condition (17) can be satisfied if and only if the condition (16) holds. ■

Theorem 2. The any fractional discrete-time system (3) is positive if and only if

$$0 < \alpha_j \le 1 \quad \text{for} \quad j = 1, \ldots, q, \quad q \le n, \tag{19a}$$

$$A_\alpha = A + \alpha \in \Re_+^{n\times n}, \tag{19b}$$

$$B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}. \tag{19c}$$

where matrix A_α is given by (9a) or (11a).

Proof. Sufficiency: If the condition (19b) is satisfied then by Lemma 2 $\Phi_i \in \Re_+^{n\times n}$ holds for $i = 0, 1, 2, \ldots$ If (17) and (19c) are satisfied then from (9) and (3b) we have $x(i) \in \Re_+^n$ and $y(i) \in \Re_+^p$ for every $i \in Z_+$ since $x_0 \in \Re_+^n$ and $u(i) \in \Re_+^m$, $i \in Z_+$.

Necessity: Let $u_i = 0$ for $i \in Z_+$. Assuming that the system is positive from (9) for $i = 0$ we obtain $x(1) = A_\alpha x_0$ and from (3b) we have $y(0) = Cx_0 \in \Re_+^p$. This implies $A_\alpha \in R_+^{n\times n}$ and $C \in \Re_+^{p\times n}$ since $x_0 \in \Re_+^n$. by definition 1 is arbitrary. Assuming $x_0 = 0$ from (9) for $i = 0$ we obtain $x(1) = Bu(0) \in \Re_+^n$ and from (3b) we have $y(0) = Du(0) \in \Re_+^p$ which implies $B \in \Re_+^{N\times m}$ and $D \in \Re_+^{p\times m}$, since $u(0) \in \Re_+^m$ by Definition 1 is arbitrary. ■

3. Reachability of the positive fractional systems

Let e_i, $i = 1, 2, \ldots, n$, be the ith column of the identity matrix I. A column ae_i for $a > 0$ is called the monomial column, i.e. its one component is positive and the remaining components are zero.

Taking into account papers [3, 6, 8, 17] we may formulate the following definition of reachability of the positive any fractional-order system.

Definition 2. The positive any fractional-order system (3) is called reachable if for every state $x_f \in \Re_+^n$ there exists a natural number N and an input sequence $u(i) \in \Re_+^m$, $i = 0,1,2,\ldots N-1$, which steers the state of the system (3) from zero initial state (5) (i.e. $x_0 = 0$) to the desired final state $x_f \in \Re_+^n$.

Theorem 3. The positive any fractional-order system (3) for $0 < \alpha_j \le 1$, $j = 1,\ldots,q$, $q \le n$, is reachable in N steps if and only if the reachability matrix

$$R_N := [B, \Phi_1 B, \ldots, \Phi_{N-1}B] \tag{20}$$

contains N linearly independent monomial columns.

Proof. The solution of equation (3a) has the form (12). For zero initial condition $x_0 = 0$ and $i = N$ we have

$$x_f = x_N = \sum_{j=0}^{N-1} \Phi_{N-j-1} B u_j = R_N u_0^N, \tag{21}$$

where the reachability matrix has the form (20) and an input sequence has the following form

$$u_0^q = \begin{bmatrix} u(q-1) \\ u(q-2) \\ \vdots \\ u(0) \end{bmatrix}. \tag{22}$$

From Definition 2 and (21) it follows that for every $x_f \in \Re_+^n$ there exists an input sequence $u(i) \in \Re_+^m$, $i = 0,1,2,\ldots N-1$, if and only if the matrix R_N (20) contains N linearly independent monomial columns. ■

Theorem 4. The positive non-commensurate fractional-order system (3) for $0 < \alpha_j < 1$, $j = 1,\ldots,q$, $q \le n$, is reachable in N steps only if the matrix

$$[B, (A + \alpha)B] \tag{23}$$

contains N linearly independent monomial columns.

Proof. From (18) it follows that only the matrices $B \in \Re_+^{n\times m}$ and $\Phi_1 B \in \Re_+^{n\times m}$ may contain linearly independent monomial columns.

This is due to the nature of the elements Φ_i, $i = 2, 3, \ldots$ (13) which build up the reachability matrix (20) and which exhibit the particularity of being time-varying, in the sense that they are composed of nonzero diagonal matrix A_k, $k = 2, 3, \ldots$ (9c). ■

Remark 1. From Theorem 3 and 4 it follows that if a final state cannot be reached in $N = 2$ steps, then it is not reachable at all.

If the fractional system (3) is reachable and $R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} \in \Re_+^{Nm\times n}$ then the nonnegative input vector (22) which steers the state of the system (3) from zero initial state (5) (i.e. $x_0 = 0$) to the desired final state $x_f \in \Re_+^n$ is given by the formula [1, 6]

$$u_0^N = R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} x_f. \tag{24}$$

4. Controllabilty of the positive fractional systems

Taking into account papers [3, 6, 8, 17] we may formulate the following definitions of controllability of the positive any fractional-order system.

Definition 3. The positive any fractional-order system (3) is called controllable to zero in $N>0$ steps if for any nonzero initial state $x_0 \in \Re_+^N$ there exists an input sequence $u(i) \in \Re_+^m$, $i = 0, 1, \dots, N-1$, which steers the state of the system from nonzero initial condition (5) to zero ($x_f = 0$).

Definition 4. The positive any fractional-order system (3) is called controllable in $N>0$ steps if for any nonzero initial state $x_0 \in \Re_+^N$ there exists an input sequence $u(i) \in \Re_+^m$, $i = 0, 1, \dots, N-1$, which steers the state of the system from nonzero initial condition (5) to the desired final state $x_f \in \Re_+^N$.

Theorem 5. The positive any fractional-order system (3) for $0 < \alpha_j \le 1$, $j = 1, \dots, q$, $q \le n$, is controllable to zero in $N>0$ steps if and only if

$$\Phi_N = 0. \tag{25}$$

Moreover $u_i = 0$ for $i = 0, 1, \dots, N-1$.

Proof. From equation (12) for $x_f = 0$ and $i = N$ we have

$$0 = \Phi_N x_0 + R_N u_0^N, \tag{26}$$

where the matrix R_N has the form (20) and u_0^N is defined by (22).

It is well known that for finite N and $A + \alpha \in \Re_+^{n\times n}$, $\Phi_i \in \Re_+^{n\times n}$, $x_0 \in \Re_+^n$, $R_N \in \Re_+^{n\times Nm}$ do not exist positive $u_0^N \in \Re_+^{Nm}$ satisfying equation (26).

The equation (26) is satisfied for any nonzero initial condition (5) and $R_N \in \Re_+^{n\times Nm}$ if and only if the condition (25) holds and $u_0^N = 0$. ■

Theorem 6. The positive any fractional-order system (3) for $0 < \alpha_j \le 1$, $j = 1, \dots, q$, $q \le n$, is controllable to zero:

a) in $N = 1$ step if and only if

$$A_\alpha = A + \alpha = 0, \tag{27}$$

b) in an infinite number of steps if and only if the system is asymptotically stable.

Proof. From (18), (9a) and (9c) it follows that the condition (25) can be satisfied if and only if the condition (27) holds and $N = 1$.

In case b) if the system is asymptotically stable then

$$\lim_{N\to\infty} \Phi_N x_0 = 0 \tag{28}$$

for every $x_0 \in \Re_+^N$. Moreover $\Phi_N \to 0$ for $N \to \infty$ and $c_k(\alpha_j) \to 0$ Hence equation (26) is satisfied for $u_0^N = 0$ and by Theorem 5 the system is controllable in an infinite number of steps. ■

Remark 2. From formula (9b) it follows that the condition (27) can be satisfied if and only if the matrix A (4d) is the diagonal matrix.

Theorem 7. The positive any fractional-order system (3) for $0 < \alpha_j \le 1$, $j = 1, \dots, q$, $q \le n$, is controllable in $N>0$ steps only if

$$x_f - \Phi_N x_0 \in \Re_+^n \tag{29}$$

and the reachability matrix R_N (20) contains N linearly independent monomial columns.

Proof. From equation (12) for $x_f \in \Re_+^n$ and $i = N$ we have

$$x_f - \Phi_N x_0 = R_N u_0^N, \tag{30}$$

where the matrix R_N has the form (20) and u_0^N is defined by (22).

It is well known that in the case $x_f - \Phi_N x_0 \notin \Re_+^n$ does not exist positive $u_0^N \in \Re_+^{Nm}$ satisfying equation (30).

From Definition 4 and (30) it follows that if the condition (29) holds there exists an input sequence $u(i) \in \Re_+^m$, $i = 0,1, 2, \dots N-1$, if and only if the matrix R_N (20) contains N linearly independent monomial columns. ■

Moreover, if the condition

$$R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} \in \Re_+^{Nm\times n} \tag{31}$$

holds then the sequence of controls $u(i) \in \Re_+^m$, $i = 0, 1, \dots, N-1$, that transfers the system (3) from nonzero initial condition (5) to the desired final state $x_f \in \Re_+^N$ can be computed from

$$u_0^N = R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1}(x_f - \Phi_N x_0). \tag{32}$$

5. Example

Test reachability and controllability of positive non-commensurate fractional system (3) with the matrices

$$A = \begin{bmatrix} -0.5 & 0.3 \\ 0 & -0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{33}$$

and $\alpha_1 = 0.5$, $\alpha_2 = 0.6$.

The system (3) with the matrices (33) is the positive system, since

$$A_\alpha = A + \alpha = \begin{bmatrix} 0 & 0.3 \\ 0 & 0 \end{bmatrix} \in \Re_+^{2\times 2}. \tag{34}$$

Using (20) for $N = 2$ we obtain the reachability matrix

$$R_2 = [B, \Phi_1 B] = \begin{bmatrix} 0 & 0.3 \\ 1 & 0 \end{bmatrix} \tag{35}$$

which contains two linearly independent monomial columns. Therefore, by Theorem 3 the positive non-commensurate fractional-order system is reachable in two steps.

Computing u_0^2 from (24) for the final state $x_f = [1 \quad 2]^T$ we obtain

$$u_0^2 = R_2^{-1} x_f = \begin{bmatrix} 0 & 0.3 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 10/3 \end{bmatrix}. \tag{36}$$

We check out received result. Using (12) for matrices (33) with the input sequence $u(0) = 10/3$ and $u(1) = 2$ we obtain

$$x(1) = Bu(0) = \begin{bmatrix} 0 \\ 10/3 \end{bmatrix}, \tag{37a}$$

$$x(2) = A_\alpha x(1) + Bu(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \quad (37b)$$

Next, we test the controllability to zero of this system. From (34) it follows that the case a) of Theorem 6 is not satisfied. Therefore, the positive system is not controllable to zero in one step.

Using (13) for $i = 1, 2, \ldots$ we obtain fundamental matrices Φ_i of the forms:

$$\Phi_1 = A + \alpha = \begin{bmatrix} 0 & 0.3 \\ 0 & 0 \end{bmatrix},$$

$$\Phi_2 = A_2 = \begin{bmatrix} 0.125 & 0 \\ 0 & 0.12 \end{bmatrix} \text{ because, } A_\alpha^2 = 0$$

$$\Phi_3 = A_\alpha A_2 + A_2 A_\alpha + A_3 = \begin{bmatrix} 0.0625 & 0.0735 \\ 0 & 0.0560 \end{bmatrix} \quad (38)$$

$$\Phi_4 = \begin{bmatrix} 0.0547 & 0.0355 \\ 0 & 0.0480 \end{bmatrix},$$

$$\vdots$$

From the above and Theorem 6 it follows that the positive system is controllable to zero in an infinite number of steps.

Next, we find the sequence of inputs that transfers this system from initial condition $x_0 = [3 \quad 1]^T$ to the final state $x_f = [1 \quad 3]^T$.

Note that the conditions (29) of Theorem 7 are satisfied because the vector

$$x_f - \Phi_2 x_0 = \begin{bmatrix} 2.8750 \\ 0.6400 \end{bmatrix} \in \Re_+^2 \quad (39)$$

is nonnegative and the reachability matrix (35) contains two linearly independent monomial columns. Therefore, the positive system is controllable in 2 steps and the sequence of controls $u(i) \in \Re_+^1$, $i = 0, 1$, computed from (32) has the form

$$u_0^2 = R_2^{\mathrm{T}} [R_2 R_2^{\mathrm{T}}]^{-1} (x_f - \Phi_2 x_0) = \begin{bmatrix} 0.6400 \\ 9.5833 \end{bmatrix} \quad (40)$$

To verify obtained result we find the solution of equation (3a) with matrices (33) and $x_0 = [1 \quad 3]^T$, $u(0) = 9.5833$, $u(1) = 0.64$.

Using (12) for $i = 0,1$ we obtain, respectively

$$x(1) = A_\alpha x_0 + Bu(0) = \begin{bmatrix} 0 & 3/10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 9.583 = \begin{bmatrix} 0.9000 \\ 9.5833 \end{bmatrix},$$

$$x(2) = A_\alpha x(1) + A_2 x_0 + Bu(1) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

6. Concluding remarks

The concept of positive system has been extended for the linear discrete-time non-commensurate fractional-order systems described by the state equations. Necessary and sufficient conditions for the positivity (Theorem 2), reachability (Theorem 3) and controllability to zero (Theorem 4) for orders of the fractional difference α_j satisfied the following conditions $0 < \alpha_j \le 1$, $j = 1, \ldots, q$, $q \le n$, have been established.

Only sufficient conditions for controllability of such a system have been given. A formula for computing a nonnegative input u_0^N (32) which steers the state of the system (3) from initial state (5) to the desired final state $x_f \in \Re_+^N$ has also been given.

The considerations can be easily extended for the positive 2D fractional linear systems.

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Osiągalność i sterowalność dodatnich układów dyskretnych niecałkowitego rzędu

Streszczenie: W pracy rozpatrzono liniowe stacjonarne dodatnie układy dyskretne niecałkowitego niewspółmiernego rzędu. Sformułowano definicje oraz podano warunki konieczne i wystarczające dodatniości, osiągalności i sterowalności układów dyskretnych niewspółmiernego rzeczywistego rzędu oraz współmiernego niecałkowitego rzędu. Rozważania zilustrowano przykładem numerycznym.

Słowa kluczowe: niecałkowity niewspółmierny rząd, układ dyskretny, standardowy, dodatni, osiągalność, sterowalność

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