prof. dr hab. inż. Tadeusz Kaczorek
Białystok University of Technology
Faculty of Electrical Engineering

# POSITIVE FRACTIONAL LINEAR SYSTEMS

*An overview of some recent published and unpublished results on positive fractional continuous-time and discrete-time linear systems is given. The first part of the paper is devoted to the positive continuous-time fractional systems. For those systems the solutions to the fractional state equations are proposed. Necessary and sufficient conditions for the positivity, reachability and stability are established. In the second part similar problems are considered for positive discrete-time fractional systems.*

## DODATNIE UKŁADY LINIOWE NIECAŁKOWITEGO RZĘDU

*W pracy dokonano syntetycznego przeglądu nowych publikowanych i niepublikowanych wyników dotyczących dodatnich ciągłych i dyskretnych układów liniowych niecałkowitego rzędu. W części pierwszej poświeconej układom ciągłym podano rozwiązanie układu równań stanu, warunki konieczne i wystarczające dodatniości, osiągalności i stabilności układów dodatnich. W części drugiej przedstawiono podobne wyniki dla układów dyskretnych.*

## 1. INTRODUCTION

A dynamical system is called positive if and only if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in monographs [4, 5]. Variety of models having positive linear behavior can be found in engineering, management sciences, economics, social sciences, biology and medicine, etc.. Mathematical fundamentals of the fractional calculus are given in the monographs [14−16]. The positive fractional linear systems have been introduced in [6, 7]. Stability of fractional linear 1D discrete-time and continuous-time systems has been investigated in the papers [1, 3, 8, 17] and of 2D fractional positive linear systems in [9]. The notion of practical stability of positive fractional discrete-time linear systems has been introduced in [10] and the positive linear systems consisting of *n* subsystems with different fractional order has been analyzed in [13]. Some recent interesting results in fractional systems theory and its applications can be found in [18−20].

In this paper an overview of some recent results on positive fractional continuous-time and discrete-time linear systems is given. Some new unpublished results are also included. The paper is organized as follows. In section 2 the solutions to the fractional state equations of continuous-time linear systems are recalled. The necessary and sufficient conditions for the internal and external positivity of the fractional continuous-time linear systems are given in section 3. The section 4 is devoted to the linear continuous-time systems described by two matrix fractional differential equations of different order and the fractional electrical circuits. The reachability of fractional standard and positive linear systems is addressed in section 5. The asymptotic stability of positive continuous-time linear systems without and with delays is considered in section 6. The fractional discrete-time linear systems are addressed in section 7. The internally and externally positive fractional discrete-time linear systems are considered in section 8 and the reachability in section 9. The asymptotic stability of positive discrete-time linear systems is analyzed in section 10 and fractional different orders discrete-time linear

systems in section 11. Some concluding remarks, extensions and open problems are given in section 12.
The following notation will be used: $\Re$ – the set of real numbers, $Z_+$ – the set of nonnegative integers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, $I_n$ – the $n\times n$ identity matrix. A real square matrix is called monomial if each its row and each its column contains only one positive entry and the remaining entries are zero.

## 2. FRACTIONAL CONTINUOUS-TIME LINEAR SYSTEMS

In this paper first of all the following Caputo definition of the fractional derivative will be used [7]

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau \tag{2.1}$$

where $n-1<\alpha<n$, $n\in N=\{1,2,...\}$

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \tag{2.2}$$

is the gamma Euler function and

$$f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n} \tag{2.3}$$

The Riemann-Liouville definition of the fractional derivative has the form [7]

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}f(\tau)d\tau \tag{2.4}$$

where $n-1<\alpha<n$, $n\in N$.
Consider the continuous-time fractional linear system described by the state equations [7]

$$D_t^\alpha x(t) = \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t)+Bu(t), \quad 0<\alpha\le 1 \tag{2.5a}$$

$$y(t) = Cx(t)+Du(t) \tag{2.5b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

*Theorem 2.1.* The solution of equation (2.5a) is given by

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \quad x(0)=x_0 \tag{2.6}$$

where

$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^\infty \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{2.7}$$

$$\Phi(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{2.8}$$

and $E_\alpha(At^\alpha)$ is the Mittage-Leffler matrix function, $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ is the gamma function.

Proof is given in [7, 6].

From (2.7) and (2.8) for $\alpha = 1$ we have $\Phi_0(t) = \Phi(t) = \sum_{k=0}^{\infty} \frac{(At)^k}{\Gamma(k+1)} = e^{At}$ .

*Example 2.1.* Find the solution of equation (2.5a) for $0 < \alpha \le 1$ and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad u(t) = 1(t) = \begin{cases} 1 \text{ for } t > 0 \\ 0 \text{ for } t < 0 \end{cases} \tag{2.9}$$

Using (2.7) and (2.8) we obtain

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} = I_2 + \frac{At^{\alpha}}{\Gamma(\alpha+1)} \tag{2.10a}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = I_2 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + A \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} \tag{2.10b}$$

since $A^k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for $k = 2,3,\ldots$.

Substitution of (2.9) and $u(t) = 1$ into (2.6) yields

$$\begin{aligned} x(t) &= \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau = x_0 + \frac{Ax_0 t^{\alpha}}{\Gamma(\alpha+1)} + \int_0^t \left( \frac{B}{\Gamma(\alpha)}(t-\tau)^{\alpha-1} + \frac{AB}{\Gamma(2\alpha)}(t-\tau)^{2\alpha-1} \right) d\tau \\ &= x_0 + \frac{Ax_0 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{Bt^{\alpha}}{\Gamma(\alpha+1)} + \frac{ABt^{2\alpha}}{\Gamma(2\alpha+1)} = \begin{bmatrix} 1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ 1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \end{bmatrix} \end{aligned} \tag{2.11}$$

since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

*Theorem 2.2.* The solution of equation (2.5a) for $n-1 < \alpha < n$ and Caputo definition has the form

$$x(t) = \sum_{l=1}^{n} \Phi_l(t) x^{(l-1)}(0^+) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau \tag{2.12}$$

where

$$\Phi_l(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k\alpha+l)-1}}{\Gamma(k\alpha+l)} \quad \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{2.13}$$

Proof is given in [6, 7].

*Theorem 2.3.* The solution of equation (2.5a) for $n-1 < \alpha < n$ and the Riemann-Liouville definition has the form

$$x(t) = \sum_{l=1}^{n} \Phi_l(t) x^{(\alpha-l)}(0^+) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau \tag{2.14}$$

where

$$\Phi_l(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-l}}{\Gamma[(k+l)\alpha-l+1]} \quad \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{2.15}$$

Proof is given in [6, 7].

From comparison of (2.12) and (2.14) it follows that the component of the solution corresponding to $u(t)$ is the same.

## 3. POSITIVITY OF THE FRACTIONAL SYSTEMS

### 3.1. Internal positivity

*Definition 3.1.* The fractional system (2.5) is called the internally positive fractional system if and only if $x(t) \in \Re_+^n$ and $y(t) \in \Re_+^p$ for $t \geq 0$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \geq 0$.

*Definition 3.2.* A square real matrix $A = [a_{ij}]$ is called the Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i \neq j$. The set $n \times n$ of Metzler matrices will be denoted by $M_n$.

*Theorem 3.1.* The continuous-time fractional system (2.5) is internally positive if and only if

$$A \in M_n, \quad B \in \Re_+^{n \times m}, \quad C \in \Re_+^{p \times n}, \quad D \in \Re_+^{p \times m} \tag{3.1}$$

Proof is given in [7].

### 3.2. External positivity

*Definition 3.3.* The fractional system (2.5) is called externally positive if and only if $y(t) \in \Re_+^p, t \geq 0$ for every input $u(t) \in \Re_+^m, t \geq 0$ and $x_0 = 0$.

Matrix of the impulse responses $g(t)$ of the fractional system (2.5) is given by [7]

$$g(t) = C\Phi(t)B + D\delta(t) \quad \text{for} \quad t \geq 0 \tag{3.2}$$

*Theorem 3.2.* The continuous-time fractional system (2.5) is externally positive if and only if its impulse response matrix (3.2) is nonnegative, i.e.

$$g(t) \in \Re_+^{p \times m} \quad \text{for} \quad t \geq 0 \tag{3.3}$$

Proof is given in [7].

The matrix of impulse responses (3.2) of internally positive system (2.5) is nonnegative for $t \geq 0$. Between the internally and external positivity we have the following relationship. Every fractional continuous-time (internally) positive system (2.5) is always externally positive.

## 4. FRACTIONAL ELECTRICAL CIRCUITS

Consider a fractional linear system described by the equation

$$\begin{bmatrix} \dfrac{d^\alpha x_1}{dt^\alpha} \\ \dfrac{d^\beta x_2}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \; p-1 < \alpha < p; \;\; q-1 < \beta < q; \;\; p,q \in N \tag{4.1}$$

where $x_1 \in \Re^{n_1}$ and $x_2 \in \Re^{n_2}$ are the state vectors, $A_{ij} \in \Re^{n_i \times n_j}$, $B_i \in \Re^{n_i \times m}$; $i$, $j$ = 1,2, and $u \in \Re^m$ is the input vector.

Initial conditions for (4.1) have the form

$$x_1(0) = x_{10} \text{ and } x_2(0) = x_{20} \tag{4.2}$$

*Theorem 4.1.* The solution of the equation (4.1) for $0 < \alpha < 1$; $0 < \beta < 1$ with initial conditions (4.2) has the form

$$x(t) = \Phi_0(t)x_0 + \int_0^t [\Phi_1(t-\tau)B_{10} + \Phi_2(t-\tau)B_{01}]u(\tau)d\tau \tag{4.3}$$

where

$$x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix},\quad x_0=\begin{bmatrix}x_{10}\\x_{20}\end{bmatrix},\quad B_{10}=\begin{bmatrix}B_1\\0\end{bmatrix},\quad B_{01}=\begin{bmatrix}0\\B_2\end{bmatrix}$$

$$T_{kl}=\begin{cases}I_n & \text{for}\quad k=l=0\\ \begin{bmatrix}A_{11} & A_{12}\\0 & 0\end{bmatrix} & \text{for}\quad k=1, l=0\\ \begin{bmatrix}0 & 0\\A_{21} & A_{22}\end{bmatrix} & \text{for}\quad k=0, l=1\\ T_{10}T_{k-1,l}+T_{01}T_{k,l-1} & \text{for}\quad k+l>0\end{cases} \tag{4.4}$$

$$\Phi_0(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{kl}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)} \tag{4.5a}$$

$$\Phi_1(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{kl}\frac{t^{(k+1)\alpha+l\beta-1}}{\Gamma[(k+1)\alpha+l\beta]} \tag{4.5b}$$

$$\Phi_2(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{kl}\frac{t^{k\alpha+(l+1)\beta-1}}{\Gamma[k\alpha+(l+1)\beta]} \tag{4.5c}$$

Proof is given in [11].
Note that if $\alpha=\beta$ then from (4.5a) we have

$$\Phi_0\big|_{\alpha=\beta}(t)=\sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{4.6a}$$

From comparison of (4.5a) and (4.6a) and using (4.4) it is easy to show that

$$\sum_{\substack{i=0\\i+j=k}}^{k}\sum_{j=0}^{k}T_{ij}\frac{t^{i\alpha+j\beta}}{\Gamma(i\alpha+j\beta+1)}\Bigg|_{\alpha=\beta}=\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{4.6b}$$

*Definition 4.1.* The fractional system (4.1) is called positive if $x_1\in\Re_+^{n_1}$ and $x_2\in\Re_+^{n_2}$, $t\geq 0$ for any initial conditions $x_{10}\in\Re_+^{n_1}$, $x_{20}\in\Re_+^{n_2}$ and all input vectors $u\in\Re_+^{m}$, $t\geq 0$.

*Theorem 4.2.* The fractional system (4.1) for $0<\alpha<1$; $0<\beta<1$ is positive if and only if

$$A=\begin{bmatrix}A_{11} & A_{12}\\A_{21} & A_{22}\end{bmatrix}\in M_n,\quad \begin{bmatrix}B_1\\B_2\end{bmatrix}\in R_+^{n\times m},\ (n=n_1+n_2) \tag{4.7}$$

Proof is given in [11].
These considerations can be extended for the set of *p* matrix differential equations with different fractional orders [13].
Consider linear electrical circuits composed of resistors, supercondensators (ultracapacitors), coils and voltage (current) sources. As the state variables (the components of the state vector *x*) the voltage across the supercondensators and the currents in the coils are usually chosen. It is well-known that the current $i(t)$ in supercondensator with its voltage $u_C(t)$ is related by the formula

$$i_C(t)=C\frac{d^\alpha u_C(t)}{dt^\alpha}\quad\text{for } 0<\alpha<1 \tag{4.8}$$

where *C* is the capacity of the supercondensator.
Similarly, the voltage $u_L(t)$ on the coil with its current $i_L(t)$ is related by the formula

$$u_L(t) = L\frac{d^\beta i_L(t)}{dt^\beta} \text{ for } 0 < \beta < 1 \qquad (4.9)$$

where $L$ is the inductance of the coil.
Using the relations (4.8), (4.9) and the Kirchhoff's laws we may write for the fractional linear circuits the following state equation

$$\begin{bmatrix} \frac{d^\alpha x_C}{dt^\alpha} \\ \frac{d^\beta x_L}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_C \\ x_L \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e \qquad (4.10)$$

where the components of $x_C \in \Re^{n_1}$ are voltages across the supercondensators, the components of $x_L \in \Re^{n_2}$ are currents in coils and the components of $e \in \Re^m$ are the voltages of the circuit.

*Example 4.1.* Consider the linear electrical circuit shown on Fig. 4.1 with known resistances $R_1, R_2, R_3$, capacitances $C_1, C_2$, inductances $L_1, L_2$ and sources voltages $e_1, e_2$.

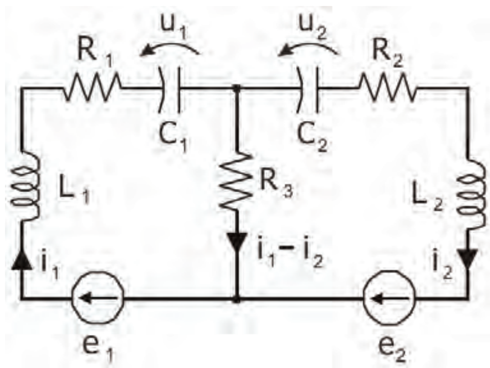


*Fig. 4.1.* Electrical circuit.

Using relations (4.8), (4.9) and the Kirchhoff's laws we may write for the circuit the following equations.

$$\begin{aligned} & i_1 = C_1\frac{d^\alpha u_1}{dt^\alpha}, \quad i_2 = C_2\frac{d^\alpha u_2}{dt^\alpha} \\ & e_1 = (R_1 + R_3)i_1 + L_1\frac{d^\beta i_1}{dt^\beta} + u_1 - R_3 i_2 \\ & e_2 = (R_2 + R_3)i_2 + L_2\frac{d^\beta i_2}{dt^\beta} + u_2 - R_3 i_1 \end{aligned} \qquad (4.11)$$

The equations (4.11) can be written in the form

$$\begin{bmatrix} \frac{d^\alpha u_1}{dt^\alpha} \\ \frac{d^\alpha u_2}{dt^\alpha} \\ \frac{d^\beta i_1}{dt^\beta} \\ \frac{d^\beta i_2}{dt^\beta} \end{bmatrix} = A\begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \qquad (4.12)$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_1} \\ -\frac{1}{L_1} & 0 & -\frac{R_1+R_3}{L_1} & -\frac{R_3}{L_1} \\ o & -\frac{1}{L_2} & -\frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} \tag{4.13}$$

From (4.13) it follows that the fractional electrical circuit is not positive since the matrix $A$ has some negative off-diagonal entries.
If the fractional linear circuit is not positive but the matrix $B$ has nonnegative entries (see for example the circuits on Fig. 4.1) then using the state-feedback

$$e = K\begin{bmatrix} x_C \\ x_L \end{bmatrix} \tag{4.14}$$

we may usually choose the gain matrix $K \in \Re^{m\times n}$ so that the closed-loop system matrix (obtained by substitution of (4.14) into (4.10))

$$A_c = A + BK \tag{4.15}$$

is a Metzler matrix.

*Theorem 4.3.* Let $A$ be not a Metzler matrix but $B \in \Re_+^{n\times m}$. Then there exists a gain matrix $K$ such that the closed-loop system matrix $A_c \in M_n$ if and only if

$$\text{rank}[B, A_c - A] = \text{rank}B \tag{4.16}$$

*Proof.* By Kronecker-Cappely theorem the equation

$$BK = A_c - A \tag{4.17}$$

has a solution $K$ for any given $B$ and $A_c - A$ if and only if the conditions (4.16) is satisfied. □

*Example 4.2.* (continuation of Example 4.1).
Let

$$A_c = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_1} \\ \frac{a_1}{L_1} & 0 & -\frac{R_1+R_3}{L_1} & \frac{a_3}{L_1} \\ o & \frac{a_2}{L_2} & \frac{a_4}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix} \text{ for } a_k \geq 0 \;\; k = 1,2,3,4 \tag{4.18}$$

In this case the condition (4.16) is satisfied since

$$\text{rank}[B, A_c - A] = \text{rank}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{L_1} & 0 & \frac{a_1+1}{L_1} & 0 & 0 & \frac{a_3+R_3}{L_1} \\ 0 & \frac{1}{L_2} & 0 & \frac{a_2+1}{L_2} & \frac{a_4+R_3}{L_2} & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} = 2 \tag{4.19}$$

The equation (4.17) has the form

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a_1+1}{L_1} & 0 & 0 & \frac{a_3+R_3}{L_1} \\ 0 & \frac{a_2+1}{L_2} & \frac{a_4+R_3}{L_2} & 0 \end{bmatrix} \tag{4.20}$$

and its solution is

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} = \begin{bmatrix} a_1+1 & 0 & 0 & a_3+R_3 \\ 0 & a_2+1 & a_4+R_3 & 0 \end{bmatrix} \tag{4.21}$$

The matrix (4.21) has nonnegative entries if $a_k \geq 0$ for $k = 1,2$ and $a_k \geq R_k$ for $k = 3,4$.
In [11] it was shown that it is not always possible to choose the gain matrix $K$ so that the two conditions are satisfied:

1) the closed-loop system matrix $A_c \in M_n$,
2) the closed-loop system is asymptotically stable.

## 5. REACHABILITY OF FRACTIONAL POSITIVE LINEAR SYSTEMS

*Definition 5.1.* A state $x_f \in \Re_+^n$ of the positive fractional system (2.5) is called reachable in time $t_f$ if there exists an input $u(t) \in \Re_+^m$, $t \in [0, t_f]$ which steers the state of the system (2.5) from zero initial state $x_0 = 0$ to the state $x_f$. If every state $x_f \in \Re_+^n$ is reachable in time $t_f$ then the system is called reachable in time $t_f$. If for every state $x_f \in \Re_+^n$ there exists a time $t_f$, such that the state is reachable in time $t_f$ then the system (2.5) is called reachable.

*Theorem 5.1.* The positive fractional system (2.5) is reachable in time $t_f$ if the matrix

$$R(t_f) = \int_0^{t_f} \Phi(\tau) B B^T \Phi^T(\tau) d\tau \tag{5.1}$$

is monomial matrix. Moreover the input which steers the state of system (2.5) from $x_0 = 0$ to $x_f$ is given by the formula

$$u(t) = B^T \Phi^T (t_f - t) R^{-1}(t_f) x_f \tag{5.2}$$

where $T$ denotes the transpose.
Proof is given in [7].

*Theorem 5.2.* If the matrix $A = diag[a_1 \quad a_2 \quad \dots \quad a_n] \in \Re_+^{n \times n}$ and $B \in \Re_+^{n \times m}$ for $m = n$ are monomial matrices then the system (2.5) is reachable.
Proof is given in [7].

*Example 5.1.* We shall show that the fractional system (2.5) with the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5.3}$$

is reachable. Taking into account that

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ for } k = 1,2,\dots \tag{5.4}$$

and using (2.8) we obtain

$$\Phi(t)=\sum_{k=0}^{\infty}\frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}=\begin{bmatrix}\Phi_1(t) & 0\\ 0 & \Phi_2(t)\end{bmatrix} \tag{5.5}$$

where

$$\Phi_1(t)=\sum_{k=0}^{\infty}\frac{t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]},\quad \Phi_2(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)} \tag{5.6}$$

and

$$\Phi(t)B=\begin{bmatrix}0 & \Phi_1(t)\\ \Phi_2(t) & 0\end{bmatrix} \tag{5.7}$$

In this case we have

$$R(t_f)=\int_0^{t_f}\Phi(\tau)B[\Phi(\tau)B]^T d\tau=\int_0^{t_f}\begin{bmatrix}\Phi_1^2(\tau) & 0\\ 0 & \Phi_2^2(\tau)\end{bmatrix}d\tau \tag{5.8}$$

The matrix (5.8) is monomial and by Theorem 3.2 the fractional system is reachable.
The considerations can be extended for positive continuous-time linear systems with delays.
Consider the continuous-time linear system with $q$ delays in state

$$\dot{x}(t)=A_0x(t)+\sum_{k=1}^{q}A_kx(t-d_k)+Bu(t) \tag{5.9a}$$

$$y(t)=Cx(t)+Du(t) \tag{5.9b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$ and $y(t)\in\Re^p$ are the state, input and output vectors $A_k, k=0,1,...,q;$ $B,C,D$ are real matrices of appropriate dimensions and $d_k$ $k=1,2,...,q$ is a delay ( $d_k\geq 0$ ).
The initial conditions for (5.9a) has the form

$$x(t)=x_0(t)\ \text{for}\ t\in[-d,0],\ d=\max_k d_k \tag{5.10}$$

where $x_0(t)\in\Re^n$ is a given.

*Definition 5.2.* the system (5.9) is called (internally) positive if and only if $x(t)\in\Re_+^n$, $y(t)\in\Re_+^p$ for any $x_0(t)\in\Re_+^n$ and for all inputs $u(t)\in\Re_+^m$, $t\geq 0$.

*Theorem 5.3.* The system (5.9) is (internally) positive if and only if

$$A_0\in M_n,\quad A_k\in\Re_+^{n\times n}, k=1,...,q\quad B\in\Re_+^{n\times m},\quad C\in\Re_+^{p\times n},\quad D\in\Re_+^{p\times m} \tag{5.11}$$

Proof is given in [7].

## 6. ASYMPTOTIC STABILITY OF POSITIVE CONTIUNUOUS-TIME LINEAR SYSTEMS

Consider the autonomous continuous-time linear system

$$\dot{x}(t)=Ax(t) \tag{6.1}$$

where $x(t)\in\Re^n$ is state vector and $A=[a_{ij}]\in\Re^{n\times n}$.

The system (6.1) is called (internally) positive if $x(t)\in\Re_+^n$, $t\geq 0$ for any initial conditions $x(0)\in\Re_+^n$. The system (6.1) is positive if and only if $A$ is a Metzler matrix [5]. It is assumed that all diagonal entries $a_{ii}$, $i=1,...,n$ of the Metzler matrix are negative, otherwise the positive system (6.1) is unstable [5].

*Theorem 6.1*. The matrix $A \in \Re^{n\times n}$ is a asymptotically stable Metzler matrix if and only if one of the following equivalent conditions is satisfied:

i) all coefficients $a_0,...,a_{n-1}$ of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{6.2}$$

are positive, i.e. $a_i \geq 0$, $i = 0,1,\ldots,n-1$

ii) all principal minors $M_i$, $i = 1,...,n$ of the matrix $-A$ are positive, i.e.

$$M_1 = |-a_{11}| > 0,\quad M_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0,\quad ..., M_n = \det[-A] > 0 \tag{6.3}$$

iii) the diagonal entries of the matrices

$$A_{n-k}^{(k)} \text{ for } k = 1,\ldots,n-1 \tag{6.4}$$

are negative, where

$$A_n^{(0)} = A = \begin{bmatrix} a_{11}^{(0)} & \ldots & a_{1,n}^{(0)} \\ \vdots & \ldots & \vdots \\ a_{n,1}^{(0)} & \ldots & a_{n,n}^{(0)} \end{bmatrix} = \begin{bmatrix} A_{n-1}^{(0)} & b_{n-1}^{(0)} \\ c_{n-1}^{(0)} & a_{n,n}^{(0)} \end{bmatrix},\quad A_{n-1}^{(0)} = \begin{bmatrix} a_{11}^{(0)} & \ldots & a_{1,n-1}^{(0)} \\ \vdots & \ldots & \vdots \\ a_{n-1,1}^{(0)} & \ldots & a_{n-1,n-1}^{(0)} \end{bmatrix}$$

$$b_{n-1}^{(0)} = \begin{bmatrix} a_{1,n}^{(0)} \\ \vdots \\ a_{n-1,n}^{(0)} \end{bmatrix},\quad c_{n-1}^{(0)} = [a_{n,1}^{(0)} \quad \ldots \quad a_{n,n-1}^{(0)}]$$

$$A_{n-k}^{(k)} = A_{n-k}^{(n-1)} - \frac{b_{n-k}^{(k-1)} c_{n-k}^{(k-1)}}{a_{n-k+1,n-k+1}^{(k-1)}} = \begin{bmatrix} a_{11}^{(k)} & \ldots & a_{1,n-k}^{(k)} \\ \vdots & \ldots & \vdots \\ a_{n-k,1}^{(k)} & \ldots & a_{n-k,n-k}^{(k)} \end{bmatrix} = \begin{bmatrix} A_{n-k-1}^{(k)} & b_{n-k-1}^{(k)} \\ c_{n-k-1}^{(k)} & a_{n-k,n-k}^{(k)} \end{bmatrix} \tag{6.5}$$

$$b_{n-k-1}^{(k)} = \begin{bmatrix} a_{1,n-k}^{(k)} \\ \vdots \\ a_{n-k-1,n-k}^{(k)} \end{bmatrix},\quad c_{n-k-1}^{(k)} = [a_{n-k,1}^{(k)} \quad \ldots \quad a_{n-k,n-k-1}^{(k)}]$$

for $k = 0,1,\ldots,n-1$.

Proof is given in [7].

*Example 6.1*. Using Theorem 6.1 check the asymptotic stability of the sportive system (6.1) with matrix

$$A = \begin{bmatrix} -0.5 & 0.1 \\ 0.2 & -0.6 \end{bmatrix}. \tag{6.6}$$

Using (6.2) we obtain

$$\det[I_n s - A] = \begin{vmatrix} s+0.5 & -0.1 \\ -0.2 & s+0.6 \end{vmatrix} = s^2 + 1.1s + 0.28 \tag{6.7}$$

all coefficient of the polynomial are positive and the condition i) is satisfied.
The condition ii) is also satisfied since

$$M_1 = 0.5,\quad M_2 = \det[-A] = \begin{vmatrix} 0.5 & -0.1 \\ -0.2 & 0.6 \end{vmatrix} = 0.28 \tag{6.8}$$

Using (6.5) for $n = 2$ we obtain

$$A_1^{(1)} = a_{11} - \frac{a_{12}a_{21}}{a_{22}} = -0.5 + \frac{0.1 \cdot 0.2}{0.6} = -\frac{0.28}{0.6} < 0 \tag{6.9}$$

Therefore, the conditions of Theorem 6.1 are met and the positive system (6.1) with (6.6) is asymptotically stable.

*Theorem 6.2.* The positive system

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{q} A_k x(t - d_k) + Bu(t) \tag{6.10a}$$

$$y(t) = CAx(t) + Du(t) \tag{6.10b}$$

is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \Re_+^n$ satisfying the condition

$$A\lambda < 0, \quad A = \sum_{k=0}^{q} A_k \tag{6.11}$$

Proof is given in [7].

*Remark 6.1.* As strictly positive vector $\lambda$ we may choose the equilibrium point

$$x_e = -A^{-1}Bu \tag{6.12}$$

since

$$A\lambda = A(-A^{-1}Bu) = -Bu < 0 \text{ for } Bu > 0 \tag{6.13}$$

*Theorem 6.3.* The positive system with delays (6.10) is asymptotically stable if and only if the positive system without delays

$$\dot{x} = Ax, \quad A = \sum_{k=0}^{q} A_k \in M_n \tag{6.14}$$

is asymptotically stable.
Proof is given in [7].

## 7. FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

*Definition 7.1.* The discrete-time function

$$\Delta^{\alpha} x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j} \tag{7.1}$$

where $0 < \alpha < 1$, $\alpha \in \Re$, and

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for} \quad k = 0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} & \text{for} \quad k = 1,2,3,... \end{cases} \tag{7.2}$$

is called the fractional $\alpha$ order difference of the function $x_k$.
The state equations of fractional discrete-time linear system have the form

$$\Delta^{\alpha} x_{k+1} = Ax_k + Bu_k, \quad u \in Z_+ \tag{7.3a}$$

$$y_k = Cx_k + Du_k \tag{7.3b}$$

where $x_k \in \Re^n$, $u_k \in \Re^m$, $y_k \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$, $D \in \Re^{p \times m}$.
Substituting the definition of fractional difference (7.1) into (7.3a), we obtain

$$x_{k+1}+\sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}x_{k-j+1}=Ax_k+Bu_k,\ k\in Z_+ \quad (7.4a)$$

or

$$x_{k+1}=Ax_k+\sum_{j=1}^{k+1}(-1)^{j+1}\binom{\alpha}{j}x_{k-j+1}+Bu_k=A_\alpha x_k+\sum_{j=2}^{k+1}(-1)^{j+1}\binom{\alpha}{j}x_{k-j+1}+Bu_k \quad (7.4b)$$

where

$$A_\alpha=A+\alpha I_n \quad (7.5)$$

From (7.4b) it follows that the fractional system is equivalent to the system with increasing number of delays. In practice it is assumed that $j$ is bounded by natural number $h$. In this case the equations (7.3) take the form

$$x_{k+1}=A_\alpha x_k+\sum_{j=1}^{h}(-1)^{j+1}\binom{\alpha}{j}x_{k-j}+Bu_k\quad k\in Z_+ \quad (7.6a)$$

$$y_k=Cx_k+Du_k \quad (7.6b)$$

The equations (7.6) describe a discrete-time linear system with $h$ delays.
Consider the fractional discrete-time linear systems with $h$ delays

$$\Delta^\alpha x_{k+1}=\sum_{r=0}^{h}(A_r x_{k-r}+B_r u_{k-r})\quad k\in Z_+ \quad (7.7a)$$

$$y_k=Cx_k+Du_k \quad (7.7b)$$

where $x_k\in\Re^n$ $u_k\in\Re^m$ $y_k\in\Re^p$ are the state, input and output vectors and $A_k\in\Re^{n\times n}$ $B_r\in\Re^{n\times m}$, $r=0,1,...,h$; $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Substituting the definition of fractional difference (7.1) into (7.7a), we obtain

$$x_{k+1}=\sum_{j=1}^{k+1}(-1)^{j+1}\binom{\alpha}{j}x_{k-j+1}+\sum_{w=0}^{h}(A_w x_{k-w}+B_w u_{k-w}),\ \ k\in Z_+ \quad (7.8)$$

If $i$ is bounded by natural number $L$ then from (7.8) we obtain

$$x_{k+1}=\sum_{j=1}^{L+1}(-1)^{j+1}\binom{\alpha}{j}x_{k-j+1}+\sum_{w=0}^{h}(A_w x_{k-w}+B_w u_{k-w}),\ \ k\in Z_+ \quad (7.9)$$

The state equations of the fractional discrete-time linear system with $h$ delays has the form

$$x_{k+1}+\sum_{j=1}^{k+1}(-1)^{j}\binom{\alpha}{j}x_{k-j+1}=\sum_{r=0}^{h}(A_r x_{k-r}+B_r u_{k-r}),\ \ k\in Z_+ \quad (7.10a)$$

$$y_k=Cx_k+Du_k,\ \ 0\le\alpha\le 1 \quad (7.10b)$$

where $x_k\in\Re^n$ $u_k\in\Re^m$ $y_k\in\Re^p$ are the state, input and output vectors and $A_r\in\Re^{n\times n}$ $B_r\in\Re^{n\times m}$, $r=0,1,...,h$; $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$, $h$ is the number of delays.

*Theorem 7.2.* The solution of the equation (7.10a) has the form

$$\begin{aligned}x_k=\Phi_k x_0+\sum_{r=0}^{h}\sum_{i=0}^{k-r-1}\Phi_{k-r-1-i}B_r u_i+\sum_{j=1}^{k+1}\sum_{l=-1}^{-j+1}(-1)^{j+1}\binom{\alpha}{j}\Phi_{k-l-j}x_l\\+\sum_{r=0}^{h}\sum_{l=-1}^{-r}\Phi_{k-r-l-1}A_r x_l+\sum_{r=0}^{h}\sum_{l=-1}^{-r}\Phi_{k-r-l-1}B_r u_l\end{aligned} \quad (7.11)$$

where

$$x_k\neq 0,\ \ u_k\neq 0,\ \ k=0,-1,...,-h \quad (7.12)$$

are initial conditions and the matrices $\Phi_k$ are determined by the equation

$$\Phi_{k+1} = \Phi_k(A_0 + \alpha I_n) + \sum_{i=2}^{k+1}(-1)^{i+1}\binom{\alpha}{i}\Phi_{k-i+1} + \sum_{i=1}^{k}\Phi_{k-i}A_i, \quad \Phi_0 = I_n \tag{7.13}$$

for $k = 0,1,\dots$ .
Proof is given in [7].
Substituting in (7.11) $h = 0$ we obtain the following theorem [7].

*Theorem 7.3.* The solution of the equation (7.4) has the form

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1}\Phi_{k-i-1}Bu_i \tag{7.14}$$

where the matrices $\Phi_k$ are determined by the equation

$$\Phi_{k+1} = \Phi_k(A + \alpha I_n) + \sum_{i=2}^{k+1}(-1)^{i+1}\binom{\alpha}{i}\Phi_{k-i+1}, \quad \Phi_0 = I_n \tag{7.15}$$

## 8. POSITIVE FRACTIONAL LINEAR SYSTEMS

In this section the necessary and sufficient conditions for the positivity of the fractional discrete-time linear system

$$x_{k+1} + \sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}x_{k-j+1} = Ax_k + Bu_k, \quad k \in Z_+ \tag{8.1a}$$

$$y_k = Cx_k + Du_k \tag{8.1b}$$

will be established, where $x_k \in \Re^n$, $u_k \in \Re^m$, $y_k \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Let $\Re_+^{n\times m}$ be the set of real $n \times m$ matrices with the nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$.

*Definition 8.1.* The system (8.1) is called the (internally) positive fractional system if and only if $x_k \in \Re_+^n$ and $y_k \in \Re_+^p$, $k \in Z_+$ for every initial conditions $x_0 \in \Re_+^n$ and all input sequences $u_k \in \Re_+^m$, $k \in Z_+$.

In [7] it has been shown that if $0 < \alpha < 1$, then

$$(-1)^{i+1}\binom{\alpha}{i} > 0, \quad i = 1,2,\dots \tag{8.2}$$

It is easy to check that if $0 < \alpha < 1$ and

$$[A + \alpha I_n] \in \Re_+^{n\times n} \tag{8.3}$$

then

$$\Phi_k \in \Re_+^{n\times n}, \quad k = 1,2,\dots \tag{8.4}$$

*Theorem 8.1.* The fractional system (8.1) is positive if and only if

$$A_\alpha = [A + \alpha I_n] \in \Re_+^{n\times n},\ B \in \Re_+^{n\times m},\ C \in \Re_+^{p\times n},\ D \in \Re_+^{p\times m}. \tag{8.5}$$

Proof is given in [7].

*Definition 8.2.* The fractional discrete-time system (7.8) witch $h$ delays is called (internally) positive if $x_k \in \Re_+^n$ and $y_k \in \Re_+^p, k \in Z_+$ for every input sequence $u_k \in \Re_+^m, k \in Z_+$ and any initial conditions $x_r \in \Re_+^{n\times n}$, $r = 0,-1,\dots,-h$.

*Theorem 8.2.* The fractional discrete-time system (7.8) witch $h$ delays is (internally) positive for $0<\alpha<1$ if and only if

$$A_r + c_{r+1} I_n \in \Re_+^{n\times n},\ c_r = (-1)^r \binom{\alpha}{r},\ B_r \in \Re_+^{n\times m},\ r = 0,-1,\ldots,-h,\ C \in \Re_+^{p\times n},\ D \in \Re_+^{p\times m}. \quad (8.6)$$

Proof is given in [7].

*Definition 8.3.* The fractional discrete-time system (8.1) is called externally positive if $y_k \in \Re_+^p, k \in Z_+$ for every input sequence $u_k \in \Re_+^m, k \in Z_+$ and $x_0 = 0$.

*Theorem 8.3.* The fractional discrete-time system (8.1) is externally positive if and only if its response matrix

$$g_k = \begin{cases} D & \text{for } k = 0 \\ CA^{k-1}B & \text{for } k = 1,2,\ldots \end{cases} \quad (8.7)$$

is nonnegative, i.e.

$$g_k \in \Re_+^{p\times m} \quad \text{for } k \in Z_+ \quad (8.8)$$

Proof is given in [7].
Every (internally) positive linear system is always externally positive.

*Example 8.1.* Consider the fractional system (8.1) for $0<\alpha<1$ with matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ (N=2) \quad (8.9)$$

The fractional system is positive since

$$A + I_n\alpha = \begin{bmatrix} 1+\alpha & 0 \\ 0 & 0 \end{bmatrix} \in \Re_+^{2\times 2} \quad (8.10)$$

Using (7.15) for $k = 0,1,\ldots$ we obtain

$$\Phi_1 = (A + I_n\alpha)\Phi_0 = \begin{bmatrix} 1+\alpha & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Phi_2 = (A + I_n\alpha)\Phi_1 - \binom{\alpha}{2}\Phi_0 = \begin{bmatrix} \frac{\alpha^2+5\alpha+2}{2} & 0 \\ 0 & \frac{(1-\alpha)v}{2} \end{bmatrix},$$

$$\Phi_3 = (A + I_n\alpha)\Phi_2 - \binom{\alpha}{2}\Phi_1 + \binom{\alpha}{3}\Phi_0 =$$

$$= \begin{bmatrix} \frac{3(\alpha^2+5\alpha+2)(\alpha+1) - \alpha(\alpha-1)(2\alpha+5)}{6} & 0 \\ 0 & \frac{\alpha(1-\alpha)(\alpha-2)}{2} \end{bmatrix}, \ldots \quad (8.11)$$

From (7.14) and (7.15) we have

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i \quad (8.12)$$

where $\Phi_k$ is defined by (8.11).

# 9. REACHABILITY OF STANDARD AND POSITIVE FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

## 9.1. Standard systems

*Definition 9.1.* A state $x_f \in \Re_+^n$ is called reachable in $q$ steps if there exist an input sequence $u_k \in \Re_+^m$, $k = 0,1,\dots,q-1$ which steers the state of the system (8.1) from zero $(x_0 = 0)$ to the final state $x_f$, i.e. $x_q = x_f$. If every given $x_f \in \Re_+^n$ is reachable in $q$ steps then the system (8.1) is called reachable in $q$ steps. If for every $x_f \in \Re_+^n$ there exists a number $q$ of steps such that the system is reachable in $q$ steps then the system is called reachable.

*Theorem 9.1.* The fractional system (8.1) is reachable in $q$ steps if and only if

$$\mathrm{rank}[B \quad \Phi_1 B \quad \dots \quad \Phi_{q-1}B] = n \tag{9.1}$$

Proof is given in [7].

*Theorem 9.2.* In the condition (9.1) the matrices $\Phi_1,\dots,\Phi_{q-1}$ can be substituted by the matrices $A_\alpha,\dots,A_\alpha^{q-1}$ i.e.

$$\mathrm{rank}[B \quad \Phi_1 B \quad \dots \quad \Phi_{q-1}B] = \mathrm{rank}[B \quad A_\alpha B \quad \dots \quad A_\alpha^{q-1}B] = n \tag{9.2}$$

Proof is given in [7].

*Theorem 9.3.* The fractional system (8.1) is reachable if and only if one of the equivalent conditions is satisfied:

i) The matrix $[I_n z - A_\alpha \quad B]$ has full rank i.e.

$$\mathrm{rank}[I_n z - A_\alpha \quad B] = n, \quad \forall z \in C \tag{9.3}$$

ii) The matrices $[I_n z - A_\alpha]$, $B$ are relatively left prime or equivalently it is possible using elementary column operations $(R)$ to reduce the matrix $[I_n z - A_\alpha \quad B]$ to the form $[I_n \quad 0]$ i.e.

$$[I_n z - A_\alpha \quad B] \xrightarrow{R} [I_n \quad 0]. \tag{9.4}$$

Proof is given in [7].

*Example 9.1.* Using (9.2), (9.3) and (9.4) check the reachability of the system with the matrices

$$A_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{9.5}$$

From (9.2) for $n = 3$ we have

$$\mathrm{rank}[B \quad A_\alpha B \quad A_\alpha^2 B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix} = 3. \tag{9.6}$$

By Theorem 9.2 the pair $(A_\alpha, B)$ is reachable.

From (9.3) we have

$$\mathrm{rank}[I_n z - A_\alpha \quad B] = \mathrm{rank}\begin{bmatrix} z & -1 & 0 & 0 \\ 0 & z & -1 & 0 \\ 1 & 2 & z+3 & 1 \end{bmatrix} = 3, \quad \forall z \in C \tag{9.7}$$

Using the elementary column operations we shall show that the matrix $[I_n z - A_\alpha]$ and $B$ are relatively left prime

$$\begin{bmatrix} z & -1 & 0 & 0 \\ 0 & z & -1 & 0 \\ 1 & 2 & z+3 & 1 \end{bmatrix} \xrightarrow[R[1+4\times(-1)]]{R[3+4\times(-z-3)] \; R[2+4\times(-2)]} \begin{bmatrix} z & -1 & 0 & 0 \\ 0 & z & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\xrightarrow{R[2+3\times(z)]} \begin{bmatrix} z & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R[3\times(-1)]]{R[1+2\times(z)] \; R[2\times(-)]} [0 \quad I_3] \tag{9.8}$$

Therefore, by Theorem 9.3 the pair $(A_\alpha, B)$ is reachable.

The fractional system is reachable only if the matrix $(A_\alpha, B)$ has $n$ linearly independent columns.

## 9.2. Positive systems

*Definition 9.2.* A state $x_f \in \Re_+^n$ of the positive fractional system (8.1) is called reachable in $q$ steps if there exist an input sequence $u_k \in \Re_+^m$, $k = 0,1,\ldots,q-1$ which steers the state from zero $(x_0 = 0)$ to the final state $x_f$, i.e. $x_q = x_f$. If every given $x_f \in \Re_+^n$ is reachable in $q$ steps then the positive system (8.1) is called reachable in $q$ steps. If for every $x_f \in \Re_+^n$ there exists a number $q$ of steps such that the system is reachable in $q$ steps then the system (8.1) is called reachable.

The inverse matrix of real matrix witch nonnegative entries has nonnegative entries if and only if it is a monomial matrix. The inverse matrix of monomial matrix can be found by its transposition and replacing each element of the transpose matrix by its inverse.

*Theorem 9.4.* The positive fractional system (8.1) is reachable in $q$ steps if and only if

$$R_q = [B \quad \Phi_1 B \quad \ldots \quad \Phi_{q-1} B] \tag{9.9}$$

contains $n$ linearly independent monomial columns.

Proof is given in [7].

The matrix (9.9) can not be substituted by the matrix

$$\bar{R}_q = [B \quad A_\alpha B \quad \ldots \quad A_\alpha^{q-1} B] \tag{9.10}$$

since for positive fractional systems the matrices in general case have different number o linearly independent monomial columns.

*Example 9.2.* Consider the fractional positive system (8.1) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -\alpha & 1 \\ 1 & 0 & -\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{9.11}$$

In this case

$$[A + \alpha I_n] = \begin{bmatrix} \alpha & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in \Re_+^n \tag{9.12}$$

and the matrix (9.10) for $q$ = 3 has the form

$$\bar{R}_q = [B \quad A_\alpha B \quad A_\alpha^2 B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{9.13}$$

and it contains three linearly independent monomial columns but the matrix

$$R_q = [B \quad \Phi_1 B \quad \Phi_2 B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \frac{\alpha(\alpha-1)}{2} \end{bmatrix} \tag{9.14}$$

contains only two linearly independent monomial columns.

*Theorem 9.5.* The positive fractional system (8.1) is reachable only if the matrix

$$[A + \alpha I_n \quad B] \tag{9.15}$$

contains $n$ linearly independent monomial columns.
Proof is given in [7].

*Example 9.3.* Consider the fractional system (8.1) with the matrices (9.11). Using (9.9) we obtain

$$R_2 = [B \quad \Phi_1 B] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{9.16}$$

which has only one monomial column. By Theorem 9.4 the system with (9.11) is unreachable. However using (9.15), we obtain matrix

$$[A + \alpha I_n \quad B] = \begin{bmatrix} 1+\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.17}$$

which has two linearly independent monomial columns.

*Theorem 9.6.* The positive fractional system (8.1) is reachable only if the matrix

$$[B \quad (A + \alpha I_n)B] \tag{9.18}$$

contains $n$ linearly independent monomial columns
Proof is given in [7].

## 10. ASYMPTOTIC STABILITY OF POSITIVE DISCRETE-TIME LINEAR SYSTEMS

Consider the positive linear discrete-time system

$$x_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ \tag{10.1a}$$

$$y_i = Cx_i + Du_i \tag{10.1b}$$

where, $x_i \in \Re_+^n, u_i \in \Re_+^m, y_i \in \Re_+^p$ are the state, input and output vectors and, $A \in \Re_+^{n\times n}$, $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$, $D \in \Re_+^{p\times m}$.

The positive system (10.1) is called asymptotically stable if the solution

$$x_i = A^i x_0 \tag{10.2}$$

of the equation

$$x_{i+1} = Ax_i, \; A \in \Re_+^{n\times n}, \; i \in Z_+ \tag{10.3}$$

satisfies the condition

$$\lim_{i\to\infty} x_i = 0 \quad \text{for every } x_0 \in \Re_+^n \tag{10.4}$$

*Theorem 10.1*. The positive system (10.3) is asymptotically stable if and only if one of the following conditions is satisfied:

1) All eigenvalues $z_1, z_2, \dots, z_n$ of the matrix $A$ satisfy the condition $|z_k| < 1$ for $k = 1, \dots, n$;
2) $\det[I_n z - A] \neq 0$ for $|z| \geq 1$;
3) $\rho(A) < 1$ where $\rho(A)$ is the spectral radius of the matrix $A$ defined by $\rho(A) = \max_{1 \leq k \leq n} \{|z_k|\}$;
4) All coefficients $\hat{a}_i$, $i = 0,1,...,n-1$ of the characteristic polynomial

$$p_A(z) = \det[I_n(z+1) - A] = z^n + \hat{a}_{n-1} z^{n-1} + ... + \hat{a}_1 z + \hat{a}_0 \tag{10.5}$$

are positive;

5) All principal minors of the matrix

$$\bar{A} = I_n - A = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} \\ \bar{a}_{21} & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n1} & \bar{a}_{n2} & \cdots & \bar{a}_{nn} \end{bmatrix} \tag{10.6}$$

are positive, i.e.

$$|\bar{a}_{11}| > 0, \begin{vmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{vmatrix} > 0, ..., \det \bar{A} > 0 \tag{10.7}$$

6) There exists a strictly positive vector $\bar{x} > 0$ (all components are positive) such that

$$[A - I_n]\bar{x} < 0 \tag{10.8}$$

7) All diagonal entries of the matrices $A_{n-k}^{(k)}$ for $k = 1,\dots,n-1$ are negative where the matrices $A_{n-k}^{(k)}$ are defined as follows

$$A_n^{(0)} = A - I_n = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1,n}^{(0)} \\ \vdots & \dots & \vdots \\ a_{n,1}^{(0)} & \dots & a_{n,n}^{(0)} \end{bmatrix} = \begin{bmatrix} A_{n-1}^{(0)} & b_{n-1}^{(0)} \\ c_{n-1}^{(0)} & a_{n,n}^{(0)} \end{bmatrix}, \quad A_{n-1}^{(0)} = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1,n-1}^{(0)} \\ \vdots & \dots & \vdots \\ a_{n-1,1}^{(0)} & \dots & a_{n-1,n-1}^{(0)} \end{bmatrix}$$

$$b_{n-1}^{(0)} = \begin{bmatrix} a_{1,n}^{(0)} \\ \vdots \\ a_{n-1,n}^{(0)} \end{bmatrix}, \quad c_{n-1}^{(0)} = [a_{n,1}^{(0)} \quad \dots \quad a_{n,n-1}^{(0)}] \tag{10.9a}$$

for $k = 0,1,\dots,n-1$ and

$$A_{n-k}^{(k)} = A_{n-k}^{(n-1)} - \frac{b_{n-k}^{(k-1)} c_{n-k}^{(k-1)}}{a_{n-k+1,n-k+1}^{(k-1)}} = \begin{bmatrix} a_{11}^{(k)} & \dots & a_{1,n-k}^{(k)} \\ \vdots & \dots & \vdots \\ a_{n-k,1}^{(k)} & \dots & a_{n-k,n-k}^{(k)} \end{bmatrix} = \begin{bmatrix} A_{n-k-1}^{(k)} & b_{n-k-1}^{(k)} \\ c_{n-k-1}^{(k)} & a_{n-k,n-k}^{(k)} \end{bmatrix},$$

$$b_{n-k-1}^{(k)} = \begin{bmatrix} a_{1,n-k}^{(k)} \\ \vdots \\ a_{n-k-1,n-k}^{(k)} \end{bmatrix}, \quad c_{n-k-1}^{(k)} = [a_{n-k,1}^{(k)} \quad \dots \quad a_{n-k,n-k-1}^{(k)}] \tag{10.9b}$$

Proof is given in [7].

*Theorem 10.2.* The positive system (10.3) is unstable if at least one diagonal entry of the matrix $A$ is greater than 1.
Proof is given in [7].

*Example 10.1.* Using the conditions of Theorem 10.1 check the asymptotic stability of the positive system (10.3) with matrix

$$A = \begin{bmatrix} 0.1 & 0.2 & 1 \\ 0 & 0.3 & 0.5 \\ 0 & 0 & 0.4 \end{bmatrix}. \tag{10.10}$$

The matrix (10.10) has the eigenvalues $z_1 = 0.1,\ z_2 = 0.3,\ z_3 = 0.4$. The condition 1) is satisfied and the system is asymptotically stable.
The condition 2) is also satisfied since

$$\det[I_3 z - A] = \begin{vmatrix} z-0.1 & -0.2 & -1 \\ 0 & z-0.3 & -0.5 \\ 0 & 0 & z-0.4 \end{vmatrix} \neq 0 \ \text{ for } \ |z| \geq 1. \tag{10.11}$$

The spectral radius of the matrix is equal to

$$\rho(A) = \max_{1\leq k\leq 3} |z_k| = 0.4 < 1 \tag{10.12}$$

and the condition 3) is satisfied.
In this case the characteristic polynomial (10.5)

$$p_A(z) = \det[I_n(z+1) - A] = \begin{vmatrix} z+0.9 & -0.2 & -1 \\ 0 & z+0.7 & -0.5 \\ 0 & 0 & z+0.6 \end{vmatrix} \tag{10.13}$$

$$= z^3 + 2.4z^{n-1} + 1.59z + 0.378$$

and all its coefficients are positive. Therefore, the condition 4) is satisfied.
The condition 5) is also satisfied since all principal minors of the matrix

$$\overline{A} = I_3 - A = \begin{bmatrix} 0.9 & -0.2 & -1 \\ 0 & 0.7 & -0.5 \\ 0 & 0 & 0.6 \end{bmatrix} \tag{10.14}$$

are positive

$$M_1 = 0.9,\quad M_2 = \begin{vmatrix} 0.9 & -0.2 \\ 0 & 0.7 \end{vmatrix} = 0.63,\quad \det \overline{A} = 0.378 \tag{10.15}$$

As the strictly positive vector $\overline{x}$ in (10.8) we choose the equilibrium point of the system (10.1) for $Bu = i_3 = [1 \quad 1 \quad 1]^T$, i.e.

$$\overline{x} = [I_3 - A]^{-1} 1_3 = \begin{bmatrix} 0.9 & -0.2 & -1 \\ 0 & 0.7 & -0.5 \\ 0 & 0 & 0.6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{0.378} \begin{bmatrix} 1.344 \\ 0.99 \\ 0.63 \end{bmatrix}. \tag{10.16}$$

This vector satisfies the condition (10.8) since

$$[A - I_n]\overline{x} = [A - I_n][I_n - A]^{-1} 1_n = -1_n. \tag{10.17}$$

Therefore, the condition 6) is also satisfied.
In this case using (10.9) we obtain the following matrices

$$A_3^{(0)} = \begin{bmatrix} -0.9 & 0.2 & 1 \\ 0 & -0.7 & 0.5 \\ 0 & 0 & -0.6 \end{bmatrix}, \quad A_2^{(1)} = \begin{bmatrix} -0.9 & 0.2 \\ 0 & -0.7 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}\begin{bmatrix} 0 & 0 \end{bmatrix}}{0.6} = \begin{bmatrix} -0.9 & 0.2 \\ 0 & -0.7 \end{bmatrix}, \qquad (10.18)$$

$$A_1^{(2)} = -0.9 + \frac{0.2 \cdot 0}{0.7} = [-0.9].$$

All these matrices have negative diagonal entries. Therefore, the condition 7) is also satisfied and the positive system is asymptotically stable.

## 11. FRACTIONAL DIFFERENT ORDERS DISCRETE-TIME LINEAR SYSTEMS

Consider the fractional different order discrete-time linear system

$$\begin{aligned} \Delta^{\alpha} x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k) \\ \Delta^{\beta} x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k) \end{aligned} \qquad (11.1)$$

where $x_1(k) \in \Re^{n_1}$ and $x_2(k) \in \Re^{n_2}$ are the state vectors, $A_{ij} \in \Re^{n_i \times n_j}$, $B_i \in \Re^{n_i \times m}$; $i, j$ = 1,2, and $u(k) \in \Re^m$ is the input vector.

The fractional derivative of $\alpha$ order is defined by

$$\begin{aligned} \Delta^{\alpha} x(k) &= \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x(k-j) = \sum_{j=0}^{k} c_{\alpha}(j) x(k-j) \\ c_{\alpha}(j) &= (-1)^j \binom{\alpha}{j} = (-1)^j \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} \end{aligned} \qquad (11.2)$$

Using (11.2) we can write the equation (11.1) in the form

$$\begin{aligned} x_1(k+1) &= A_{1\alpha}x_1(k) + A_{12}x_2(k) - \sum_{j=2}^{k+1} c_{\alpha}(j) x_1(k-j+1) + B_1u(k) \\ x_2(k+1) &= A_{21}x_1(k) + A_{2\beta}x_2(k) - \sum_{j=2}^{k+1} c_{\beta}(j) x_2(k-j+1) + B_2u(k) \end{aligned} \qquad (11.3)$$

*Theorem 11.1.* The solution to the fractional equation (10.1) with initial conditions $x_1(0) = x_{10}$, $x_2(0) = x_{20}$ is given by

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \Phi_k \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \sum_{i=0}^{k-1} \Phi_{k-i-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i) \qquad (11.4)$$

where $\Phi_k$ is defined by

$$\Phi_i = \begin{cases} I_{n_i} \quad (n = n_1 + n_2) & \text{for} \quad i = 0 \\ A\Phi_{i-1} - D_1\Phi_{i-2} - \dots - D_{i-1}\Phi_0 & \text{for} \quad i = 1,2,\dots,k \\ A\Phi_{i-1} - D_1\Phi_{i-2} - \dots - D_k\Phi_{i-k-1} & \text{for} \quad i = k+1, l+2, \dots \end{cases} \qquad (11.5)$$

Consider the fractional different orders discrete-time linear systems described by the equation (10.1) and

$$y(k) = C \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + Du(k) \qquad (11.6)$$

where $x_1(k) \in \Re^{n_1}$, $x_2(k) \in \Re^{n_2}$, $u(k) \in \Re^m$, $y(k) \in \Re^p$ are the state, input and output vectors, $C \in \Re^{p \times n}$, $D \in \Re^{p \times m}$.

*Definition 11.1.* The fractional system (11.1), (11.6) is called positive if and only if $x_1(k) \in \Re_+^{n_1}$, $x_2(k) \in \Re_+^{n_2}$ and $y(k) \in \Re_+^{p}$, $k \in Z_+$ for any initial conditions $x_1(0) = x_{10} \in \Re_+^{n_1}$, $x_2(0) = x_{20} \in \Re_+^{n_2}$ and all input sequences $u(k) \in \Re^m$, $k \in Z_+$.

*Theorem 11.2.* The fractional discrete-time linear system (11.1), (11,6) with $0 < \alpha < 1$, $0 < \beta < 1$ is positive if and only if

$$A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \in \Re_+^{n\times n},\ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \Re_+^{n\times m},\ C \in \Re_+^{p\times n},\ D \in \Re_+^{p\times m}. \tag{11.7}$$

Proof is given in [7].

These considerations can be easy extended to fractional systems consisting of *n* subsystems of different fractional order [13].

## 12. CONCLUDING REMARKS

An overview of some new results on positive fractional continuous-time and discrete-time linear systems has been presented. In the first part of the paper the positivity, reachability and stability of the fractional continuous-time linear systems have been addressed. In the second part similar problems for fractional discrete-time linear system have been addressed.

The presented considerations can be extended for positive 2D linear systems and 2D continuous-discrete linear systems [5, 7]. Extensions of these considerations for fractional 2D continuous-time linear and nonlinear system are open problems.

An open problem is also an extension of these considerations for fractional positive switched linear systems.

## ACKNOWLEGMENT

This work was supported by Ministry of Science and Higher Education in Poland under work No. NN514 1939 33.

## REFERENCES

[1] Busłowicz M., Stability of linear continuous time fractional order systems with delay of the retarder type, *Bull. Pol. Acad. Sci. Tech.*, vol. 56, no. 4, 2008, pp. 319−324.

[2] Dzieliński A., Sierociuk D. and Sarwas G., Ultracapacitor parameters identification based on fractional order model, *Proc ECC'09*, Budapest 2009.

[3] Dzieliński A. and Sierociuk D., Stability of discrete fractional order state-space systems, *Journal of Vibrations and Control*, vol. 14, no. 9/10, 2008, pp. 1543−1556.

[4] Farina E. and Rinaldi S., Positive Linear Systems. Theory and Applications, J. Wiley New York 200.

[5] Kaczorek T., Positive 1D and 2D Systems, Springer-Verlag, London 2002.

[6] Kaczorek T., Fractional positive continuous-time systems and their Reachability, *Int. J. Appl. Math. Comput. Sci.*, vol. 18, no. 2, 2008, pp. 223−228.

[7] Kaczorek T., Selected Problems in Fractional Systems Theory, Springer-Verlag 2011.

[8] Kaczorek T., Stability of positive continuous-time systems with delays, *Bull. Pol. Acad. Sci. Tech.*, vol. 57, no. 4, 2009, pp. 395−398.

[9] Kaczorek T., Asymptotic stability of positive fractional 2D linear systems, *Bull. Pol. Acad. Sci. Tech.*, vol. 57, no. 3, 2009, pp. 287−292.

[10] Kaczorek T., Practical stability of positive fractional discrete-time linear systems, *Bull. Pol. Acad. Sci. Tech.*, vol. 56, no. 4, 2008, pp. 313−318.

[11] Kaczorek T., Positive linear systems with different fractional orders, *Bull. Pol. Acad. Sci. Tech.*, vol. 58, no. 3, 2010, pp. 453−458.

[12] Kaczorek T., Decomposition of the pairs (A,B) and (A,C) of positive discrete-time linear systems, *Archives of Control Sciences*, vol. 20, no. 3, 2010, pp. 341−361.
[13] Kaczorek T., Positive linear systems consisting of *n* subsystems with different fractional orders, *IEEE Trana. Circuits and Systems*, 2011 (in Press).
[14] Oldham K. B. and Squier J., The Fractional Calculus, Academic Press, New York 1974.
[15] Ostalczyk P., Epitome of the fractional calculus: Theory and its Applications in Automatics, Wydawnictwo Politechniki Łódzkiej, Łódź 2008 (in Polish).
[16] Podlubny I., Fractional Differential Equations, Academic Press, San Diego 1999.
[17] Radwan A. G., Soliman A. M., Elwakil A. S. and Sedeek A., On the stability of linear systems with fractional-order elements, *Chaos, Solitons and Fractals*, vol. 40, no. 5, 2009, pp. 2317−2328.
[18] Ruszewski A., Stability regions of closed-loop system with time delay inertial plant of fractional order and fractional order PI controller, *Bull. Pol. Acad. Sci. Tech.*, vol. 56, no. 4, 2008, pp. 329−332.
[19] Tenreiro Machado J. A. and Ramiro Barbosa S., Editors of special issue on fractional differentiation and its application, *Journal of Vibration and Control*, vol. 14, no. 9/10, 2008, pp. 1543−1556.
[20] Vinager B. M., Monje C. A. and Calderon A. J., Fractional order systems and fractional order control actions, *41th IEEE Conf. on Decision and Control*, Las Vegas NV, December 2002.